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STATE CONTROLLABILITY TECHNIQUES FOR LINEAR TIME-VARYING DISCRETE SYSTEMS

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March 1995

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13. ABSTRACT *(Maximum 200 words)*
This report describes the results of original research in the control of linear systems. Contributions which advance control theory are made in the areas of discretization, controllability, canonical transformation, state feedback, and stabilization. A new approach to control linear time-varying continuous systems which do not satisfy the property of lexicografixed controllability is provided. It involves the discretization of the continuous system and the selection of a sampling period to make the resulting discrete system lexicografixedly controllable in discrete time. Methods are developed to control and stabilize the resulting linear time-varying discrete system. A controllability theorem is introduced and its associated matrix derived. Finally, a state feedback technique used to asymptotically stabilize a subclass of linear time-varying discrete systems is developed.

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CHAPTER 1
INTRODUCTION

1.1. Overview

The main components of many control systems, such as actuators and sensors, change with time. These systems can, therefore, be modeled as linear time-varying multivariable continuous systems. Much research has already been conducted in the area of the control of such systems. Techniques exist which utilize the concepts of controllability, observability, and stabilization to transform such systems into canonical forms so that state estimation and state feedback techniques can be applied to control the system [18]-[22]. Another technique exists to transform linear multivariable systems that are minimally observable into equivalent canonical forms [5]. There is also a scheme which stabilizes controllable time-varying systems [29]. Another study found that if a time-varying system is lexicografixedly controllable, then there exists a state feedback controller [21] to stabilize the system by assigning an arbitrary set of eigenvalues to the closed-loop feedback system.

Microprocessors are used in most modern control systems. Their use depends on the techniques taught in discrete-time or computer control courses [23]. However, these techniques deal mainly with time-invariant discrete systems. The concepts of controllability, observability, and stabilization for linear time-varying multivariable discrete systems have not yet been exploited. Therefore, state estimation and state feedback techniques do not exist for such systems. The lack of research in this area is probably due to the mathematical difficulties involved in solving such systems.

The outputs of linear time-varying components, such as actuators and sensors, are typically used as inputs to computer control systems. The resulting system is modeled as a linear time-varying multivariable discrete control system. Since the techniques of state feedback and state estimation do not exist for such systems, control engineers are forced to use intuition and experience instead of theory to design these control systems.

This research considers the control problem of a linear time-varying continuous system that is not lexicografixedly controllable [20]. A state feedback technique to stabilize such systems does not currently exist. A new approach, which considers the discretization of the linear time-varying continuous system and the selection of a sampling period to make the resulting discrete system lexicografixedly controllable, is proposed. Algorithms needed to control the resulting linear time-varying discrete system are also developed.

1.2. Problem Statement

A technique to control linear time-varying multivariable continuous systems which are not lexicografixedly controllable must be developed. The following statement defines the problem considered in this report:

Given a linear time-varying multivariable continuous system that is not lexicografixedly controllable, discretize the system through the application of sample and zero-order hold devices and then develop a method to stabilize the resulting discrete system through the application of state feedback.

The following obstacles must be overcome in order to solve this problem:

1) Equations which model the resulting discrete system must be derived.

2) A controllability theorem for linear time-varying discrete systems must be developed.

3) A state feedback technique which asymptotically stabilizes the linear time-varying discrete system must be developed.

This research effort focuses on overcoming these obstacles.

1.3. Research Results

The results of this In-House Laboratory Independent Research (ILIR) program provide advances in the theory of the control of linear systems. Contributions which advance control theory are made in the areas of discretization, controllability, canonical transformation, state feedback, and stabilization. Control algorithms are developed and coded in the Maple language to form a library of Maple Symbolic Math routines which can be used in the design and simulation of physical systems.

1.4. Transition of Research

The results of this research are relevant to all military and commercial systems that use time-varying components as inputs to microprocessor-based control systems. Results will be directly transitioned into Army prototype designs for generator set controls, electrical machinery controls, engine governor controllers, electric drive systems, and robotic systems. These results will also be transitioned to the commercial and academic communities through technical publications and presentations at conferences and symposia. The principal investigator presented an invited paper documenting interim ILIR results on controllability and served as a Session Chair at the 1994 International Conference on Computers and Their

Applications [15]. Interim ILIR results on state feedback and stabilization were presented in another document [16].

1.5. Report Organization

An overview of the report is given in Chapter 1. Relevant background information pertaining to linear continuous and discrete systems is reviewed in Chapter 2. Chapter 3 is devoted to the discretization of linear time-varying continuous systems. Equations modeling the discretized system in terms of its associated continuous system are derived. A controllability theorem for the class of systems discretized in Chapter 3 is developed and its associated controllability matrix is derived in Chapter 4. These results are used in the development of an equivalence transformation and a state feedback theorem found in Chapter 5. An algorithm which asymptotically stabilizes the above class of linear time-varying discrete systems is also presented. Chapter 6 presents a summary of the report and provides recommendations for future research. Appendix A includes the Maple control algorithms used to solve the example problems in Chapters 3-5. A copy of the ISCA paper documenting interim ILIR results is included as Appendix B.

CHAPTER 2
REVIEW OF LINEAR SYSTEMS

2.1. Introduction

This chapter reviews the research underlying the development of linear continuous and linear discrete systems. Sections 2.2 and 2.3 present the mathematical models describing linear continuous and linear discrete systems and the associated concepts of controllability, canonical transformations, and stabilization. It is these models and concepts which are used as the groundwork in the solution of the problem statement presented in Chapter 1.

2.2. Linear Continuous Systems

The input-output description of a system gives the mathematical relationship between the inputs and the outputs [6]. A continuous system with p inputs and q outputs can be represented by the following figure.

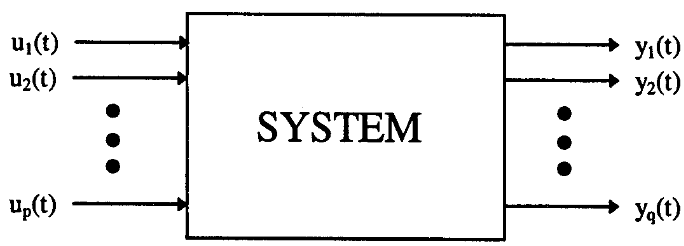

Figure 2-1 Linear Time-Varying Multivariable Continuous System

The system is said to be relaxed if there is no energy stored in the system at time $t = -\infty$. The mathematical relationship between the inputs and the outputs of a relaxed system is given by

$$y(t) = H\{u(t)\}, \tag{2-1}$$

where y(t) is the (qx1) output vector, u(t) is the (px1) input vector, and H{ } is some operator or function that uniquely specifies the output y(t) in terms of the input u(t).

A linear continuous system is one in which the principle of superposition applies [25]. That is, if $y_1(t)$ is the response to the input $u_1(t)$ and $y_2(t)$ is the response to the input $u_2(t)$ then the system is linear if and only if, for every scalar α and β, the response to the input $\alpha u_1(t)+\beta u_2(t)$ is $\alpha y_1(t)+\beta y_2(t)$. Such a system can be modeled by the following state and input equations [6],

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{2-2}$$

and

$$y(t) = C(t)x(t) + E(t)u(t), \tag{2-3}$$

where x(t) is the (nx1) state vector, A(t) is the (nxn) state matrix, B(t) is the (nxp) input matrix, C(t) is the (qxn) output matrix, and E(t) is the (qxp) direct transmission matrix.

The solution to the state equation (2-2) of the linear time-varying continuous system is given in [24] as

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau, \tag{2-4}$$

where the matrix $\Phi(t,t_0)$ is referred to as the state transition matrix. The state transition matrix is the unique solution to

$$\dot{\Phi}(t,t_0) = A(t)\Phi(t,t_0).$$

(2-5)

Equation (2-4) shows that the state transition matrix completely determines the solution of the state equation (2-2). The solution of (2-5) is very difficult, except for the following special cases.

Special Case 1: Triangular State Matrix

If the state matrix A(t) of the continuous system is a triangular matrix then the solution of (2-5) can be reduced to that of solving a set of scalar differential equations [6]. In this case the state transition matrix can be readily obtained from the following equation,

$$\Phi(t,t_0) = \Psi(t)\Psi^{-1}(t_0)$$

(2-6)

where $\Psi(t)$ is any fundamental matrix consisting of any n linearly independent solutions of

$$\dot{x}(t) = A(t)x(t).$$

(2-7)

Special Case 2: Commutative State Matrix

If the state matrix A(t) of the continuous system has the following commutative property

$$A(t)\left[\int_{t_0}^{t} A(\tau)d\tau\right] = \left[\int_{t_0}^{t} A(\tau)d\tau\right]A(t)$$

(2-8)

for all t and t_0, then the unique solution of (2-5) is given in [6] as

$$\Phi(t,t_0) = \exp\left[\int_{t_0}^{t} A(\tau)d\tau\right].$$

(2-9)

It has been shown in [6] that:

1) if A(t) is a diagonal or a constant matrix, or

2) if there is only one state

then the commutative property holds and (2-9) can be used directly.

Special Case 3: A_1 Class

If there exists a constant matrix A_1, such that the state matrix A(t) of the continuous system satisfies the following property

$$\dot{A}(t) = A_1A(t) - A(t)A_1,$$

(2-10)

then the state transition matrix can be computed by the following equation [30].

$$\Phi(t,t_0) = \exp(A_1t)\exp[A_2(t-t_0)]\exp(-A_1t_0)$$

(2-11)

Note that the constant matrix A_1 is not unique. The constant matrix A_2 is given by

$$A_2 = \exp(-A_1t_0)[A(t_0) - A_1]\exp(A_1t_0).$$

(2-12)

Special Case 4: ($h(t)$ and A_1) Class

If there exists a constant matrix A_1 and a non-zero scalar time function $h(t)$ such that the derivative of $h(t)$ exists and the state matrix $A(t)$ of the continuous system satisfies the following property

$$A_1 A(t) - A(t) A_1 = \frac{\dot{A}(t)}{h(t)} - \frac{\dot{h}(t)}{h^2(t)}, \tag{2-13}$$

then the state transition matrix can be computed by [30],

$$\Phi(t, t_0) = \exp(A_1 g(t)) \exp(A_2 g(t)) \tag{2-14}$$

where, the scalar time function $g(t)$ is given by

$$g(t) = \int_{t_0}^{t} h(\tau) d\tau . \tag{2-15}$$

Note that the constant matrix A_1 and the scalar time function $h(t)$ are not unique. The constant matrix A_2 is given by

$$A_2 = A_h(t_0) - A_1, \tag{2-16}$$

where

$$A_h(t_0) = \lim_{t \to t_0} \frac{A(t)}{h(t)}. \tag{2-17}$$

2.2.1. Controllability

The concept of controllability plays an important role in the optimal control of linear multivariable systems. If a given linear continuous system satisfies the condition of state controllability then closed-loop poles can be selected, such that the system that possesses such poles can be designed using state feedback techniques.

There are different degrees of controllability; state controllability, uniform controllability, and lexicografixed controllability. The definition of each degree is defined as follows:

Definition 2.1 *The state equation (2-2) is said to be state controllable at time t_0, if there exists a finite time $t_1>t_0$ such that for any $x(t_0)$ and $x(t_1)$ in the state space, there exists an input $u(t)$, applied from t_0 to t_1, that will transfer the state $x(t_0)$ to the state $x(t_1)$ in a finite time.*

A controllability matrix M(t) is used to test for the condition of state controllability. The controllability matrix M(t) of the continuous system given in

(2-2) is defined in[18] as

$$M(t) = \left[M_1(t) \quad M_2(t) \quad \cdots \quad M_n(t)\right], \tag{2-18}$$

where

$$M_i(t) = Z^{i-1}[B(t)] \tag{2-19}$$

for i=1,2,...,n. The differential matrix operator $Z[\cdot]$ is given in[18] as

$$Z[\cdot] = -A(t)[\cdot] + \frac{d}{dt}[\cdot],$$

(2-20)

where

$$Z^q[\cdot] \equiv \underbrace{ZZ\cdots Z}_{q \text{ times}}[\cdot]$$

(2-21)

and

$$Z^0[\cdot] \equiv [\cdot].$$

(2-22)

If the state and input matrices of the state equation are (n-1) times continuously differentiable, then the state equation is state controllable at time t_0 if there exists a time $t_1 > t_0$ such that

$$\text{rank}[M(t)] = n.$$

(2-23)

Definition 2.2 *The state equation (2-2) of a linear time-varying continuous system is said to be uniformly controllable on the time interval* $[t_0, t_1]$, *if*

$$\text{rank}[M(t)] = n$$

(2-24)

for all $t \varepsilon [t_0, t_1]$.

Definition 2.3 *A uniformly controllable system whose controllability matrix M(t) has a fixed lexicographic basis on the time interval* $[t_0, t_1]$ *is said to be lexicografixedly*

controllable on $[t_0, t_1]$. The system is lexicografixedly controllable on the time interval $[t_0, t_1]$ if there exists an (nxn) sub-matrix of M(t) that has full rank for all $t \in [t_0, t_1]$.

2.2.2. Equivalence Transformations

Transforming a linear system into an equivalent canonical form is an important technique used by control system designers. It allows the designer to work with a minimum number of system parameters [20]. An equivalence transformation is defined as follows:

Definition 2.4 *Consider the linear time-varying continuous system given by (2-2). If there exists an (nxn) matrix Q(t) which is nonsingular and continuously differentiable in t for all $t \in [t_1, t_2]$ such that $\overline{x}(t) = Q(t)x(t)$, then the system described by*

$$\dot{\overline{x}}(t) = \overline{A}(t)\overline{x}(t) + \overline{B}(t)u(t), \tag{2-25}$$

is said to be equivalent to the original system (2-2) and the matrix Q(t) is said to be an equivalence transformation on $[t_1, t_2]$.

The equivalent state and input matrices of (2-25) can be computed as

$$\overline{A}(t) = \left(Q(t)A(t) + \dot{Q}(t)\right)Q^{-1}(t) \tag{2-26}$$

and

$$\overline{B}(t) = Q(t)B(t). \tag{2-27}$$

Several techniques to transform linear systems into particular equivalent canonical forms already exist. Five such transformations for the class of linear time-varying

lexicografixedly controllable continuous systems were developed by Nguyen [18]. Since Nguyen's second canonical transformation was found to be useful in the solution of the problem statement in Chapter 1, the development of this transformation is presented in the following section.

2.2.2.1. Nguyen's Second Canonical Transformation

Nguyen's second canonical transformation [18] converts a given linear time-varying multivariable system (2-2) which satisfies the property of lexicografixed controllability into an equivalent system (2-25). The associated equivalent state and input matrices have the following canonical structure,

$$\overline{A}(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) & \cdots & A_{1p}(t) \\ A_{21}(t) & A_{22}(t) & \cdots & A_{2p}(t) \\ \vdots & \vdots & & \vdots \\ A_{p1}(t) & A_{p2}(t) & & A_{pp}(t) \end{bmatrix} \tag{2-28}$$

where

$$A_{ii}(t) = \underbrace{\left.\begin{bmatrix} X & X & \cdots & X & X \\ \hline 1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}\right\}m_i}_{m_i} \tag{2-29}$$

for i=1,2,...,p, and

$$A_{ij} = \begin{bmatrix} X_{1xm_j} \\ 0_{(m_i-1)xm_j} \end{bmatrix}$$

(2-30)

for i≠j, i,j=1,2,...,p, and

$$\overline{B}(t) = \underbrace{\begin{bmatrix} \pm1 & X & \cdots & X \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ \hdashline & & \vdots & \\ \hdashline 0 & 0 & \cdots & \pm1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{p}.$$

(2-31)

Nguyen's second canonical transformation is defined as

$$Q(t) = \begin{bmatrix} Q_1(t) \\ Q_2(t) \\ \vdots \\ Q_p(t) \end{bmatrix},$$

(2-32)

where

$$Q_i(t) = \begin{bmatrix} W^{m_i-1}[\beta_i] \\ W^{m_i-2}[\beta_i] \\ \vdots \\ [\beta_i] \end{bmatrix}$$

(2-33)

for i=1,2,...,p. The differential matrix operator $W[\cdot]$ is defined as

$$W[\cdot] = [\cdot]A(t) + \frac{d}{dt}[\cdot]. \tag{2-34}$$

The row vector β_i can be computed as follows. Let the input matrix of the given system be represented by

$$B(t) = \begin{bmatrix} b_1(t) & b_2(t) & \cdots & b_p(t) \end{bmatrix}. \tag{2-35}$$

The controllability matrix (2-18) can then be written as

$$M(t) = \begin{bmatrix} b_1(t) \cdots b_p(t) & Z[b_1(t)] \cdots Z[b_p(t)] & \cdots & Z^{n-1}[b_1(t)] \cdots Z^{n-1}[b_p(t)] \end{bmatrix}. \tag{2-36}$$

If the system is lexicografixedly controllable, then a unique set of n linearly independent column vectors of M(t) can be found. The selected vectors can be rearranged to form the following nonsingular matrix,

$$\overline{M}(t) = \begin{bmatrix} \overline{M}_1(t) & \overline{M}_2(t) & \cdots & \overline{M}_p(t) \end{bmatrix} \tag{2-37}$$

where

$$\overline{M}_i(t) = \begin{bmatrix} b_i(t) & Z[b_i(t)] & \cdots & Z^{m_i-1}[b_i(t)] \end{bmatrix} \tag{2-38}$$

for i=1,2,...,p and where the subsystem controllability index m_i is defined as the highest integer for which $Z^{m_i-1}[b_i(t)]$ is linearly independent of other column vectors of (2-37). The inverse of (2-37) can be represented as

$$\overline{M}^{-1}(t) = \begin{bmatrix} \beta_{1,0}(t) \\ \beta_{1,1}(t) \\ \vdots \\ \beta_{1,m_1-1}(t) \\ \hdashline \vdots \\ \hdashline \beta_{p,0}(t) \\ \beta_{p,1}(t) \\ \vdots \\ \beta_{p,m_p-1}(t) \end{bmatrix}, \tag{2-39}$$

where the $\beta_{i,j}$'s are the row vectors of $\overline{M}^{-1}(t)$. β_k is defined in [18] as

$$\beta_k \equiv \beta_{k,m_k-1} \tag{2-40}$$

for k=1,2,...,p.

Nguyen's second canonical transformation can now be computed by substituting (2-34) and (2-40) into (2-32) and (2-33).

2.2.2.1.1. Example

The application of Nguyen's second canonical transformation to the following linear time-varying multivariable continuous system,

$$\dot{x}(t) = \begin{bmatrix} 1 & t^3 & 0 \\ 0 & 2 & t^2 \\ t & 0 & 3 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & t^2 \end{bmatrix} u(t)$$

(2-41)

produces an equivalent system with the following canonical structure,

$$\dot{\bar{x}}(t) = \left[\begin{array}{cc:c} -3t^{-1}+3 & 6t^{-1}+3t^{-2}-2 & -t^4 \\ 1 & 0 & 0 \\ \hdashline 0 & -t^2 & 3-2t^{-1} \end{array}\right] \bar{x}(t) + \left[\begin{array}{cc} -1 & -1 \\ 0 & 0 \\ \hdashline 0 & 1 \end{array}\right] u(t).$$

(2-42)

2.2.2.1.2. Nguyen's Second Canonical Transformation for Single-Input Systems

If the given system is a linear time-varying uniformly controllable single input system with a state equation given by

$$\dot{x}(t) = A(t)x(t) + b(t)u(t),$$

(2-43)

then (2-37)-(2-40) reduce to

$$M(t) = \overline{M}(t) = \begin{bmatrix} b(t) & Z[b(t)] & \cdots & Z^{n-1}[b(t)] \end{bmatrix},$$

(2-44)

$$\overline{M}^{-1}(t) = \begin{bmatrix} \beta_{1,0}(t) \\ \beta_{1,1}(t) \\ \vdots \\ \beta_{1,n-1}(t) \end{bmatrix},$$

(2-45)

$$\beta \equiv \beta_{1,n-1},$$

(2-46)

and (2-32) reduces to

$$Q(t) = \begin{bmatrix} W^{n-1}[\beta] \\ W^{n-2}[\beta] \\ \vdots \\ [\beta] \end{bmatrix}. \tag{2-47}$$

For single-input systems, the equivalent state (2-28) and input (2-31) matrices have the following canonical form,

$$\overline{A}(t) = \underbrace{\left.\left[\begin{array}{cccc:c} X & X & \cdots & X & X \\ \hdashline 1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{array}\right]\right\} n}_{n} \tag{2-48}$$

and

$$\overline{b} = \left.\begin{bmatrix} (-1)^{n-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\} n. \tag{2-49}$$

2.2.3. Stabilization of Linear Continuous Systems

Stability is an important qualitative property in control system design. If a system is not stable, it is usually of no practical use [6]. It is well known that a time-invariant system is asymptotically stable if all of the eigenvalues have negative real parts [18]. Therefore, all

the eigenvalues of the given system must be movable so that the designer can assign them to the left hand side of the complex plane. The stability of a time-invariant continuous system can be determined directly from the state matrix A(t).

Various stability conditions have been obtained for linear time-varying continuous systems. Most of these methods are difficult to use because they depend on knowledge of the state transition matrix. However, Nguyen developed a stabilization scheme that is straight-forward and does not depend on knowledge of the state transition matrix [18]. Nguyen uses a state feedback matrix to force the closed-loop feedback equivalent system to become a time-invariant system having stable eigenvalues. This stabilization scheme uses a Lyapunov transformation to prove the asymptotic stability of a time-invariant equivalent system which implies the asymptotic stability of its corresponding time-varying original system. Nguyen's state feedback theorem for the class of linear time-varying single-input continuous systems is presented below.

If the linear time-varying single-input continuous system given by (2-43) is uniformly controllable on $[t_0,\infty)$, then there exists a state feedback law u(t)=K(t)x(t) that will asymptotically stabilize the given system (2-43). After applying the state feedback law, the given system can be modeled as

$$\dot{x}(t) = [A(t) + b(t)K(t)]x(t). \tag{2-50}$$

If K(t) is chosen properly, the resulting closed-loop system will be asymptotically stable. An algorithm for determining the required feedback matrix K(t) is given below.

Algorithm 2.1

1) Apply Nguyen's second canonical transformation to the given linear time-varying single-input continuous system. The state and input matrices of the equivalent system will assume the following form,

$$\overline{A}(t) = \left[\begin{array}{cccc} \overline{a}_{11} & \overline{a}_{12} & \cdots & \overline{a}_{1n} \\ \hline \multicolumn{3}{c|}{I_{n-1}} & 0_{(n-1)x1} \end{array}\right], \tag{2-51}$$

and

$$\overline{b} = \begin{bmatrix} (-1)^{n-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{2-52}$$

where the coefficients $\overline{a}_{1i}$ for $i = 1,2,\cdots,n$ are the time varying parameters of (2-48).

2) Select the desired eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ and compute the characteristic equation.

$$(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0 \tag{2-53}$$

3) Compute the equivalent feedback matrix $\overline{K}(t)$.

$$\overline{K}(t) = \frac{1}{(-1)^{n-1}}\left[-\overline{a}_{11} - \alpha_{n-1} \quad \cdots \quad -\overline{a}_{1n} - \alpha_0\right] \tag{2-54}$$

4) Compute the equivalent state matrix of the closed-loop feedback system.

$$\overline{A}_c = \overline{A}(t) + \overline{b}(t)\overline{K}(t) = \left[\begin{array}{cccc} -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_{01} \\ \hline & I_{n-1} & & 0_{(n-1)x1} \end{array}\right]$$

(2-55)

Computing the eigenvalues of (2-55) will verify that the equivalent system possesses the desired eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$.

5) Compute the feedback matrix K(t).

$$K(t) = \overline{K}(t)Q(t)$$

(2-56)

It is this algorithm which will serve as part of the baseline for the solution of the problem statement presented in Chapter 1.

2.3. Linear Discrete Systems

Most modern control systems are implemented through the use of sampled-data or discrete methods which rely on the use of either a microprocessor or a digital signal processor. Sample-and-hold devices are used extensively in modern control systems. A sampler converts a given analog signal into a pulse-amplitude modulated signal while a hold device simply freezes the value of the pulse for a prescribed duration of time [11]. A zero-order hold device freezes the sampled value until the next sample arrives.

If a sample and zero-order hold is applied to each input of the continuous system given in Figure 2-1, a linear time-varying multivariable discrete system as depicted below is obtained.

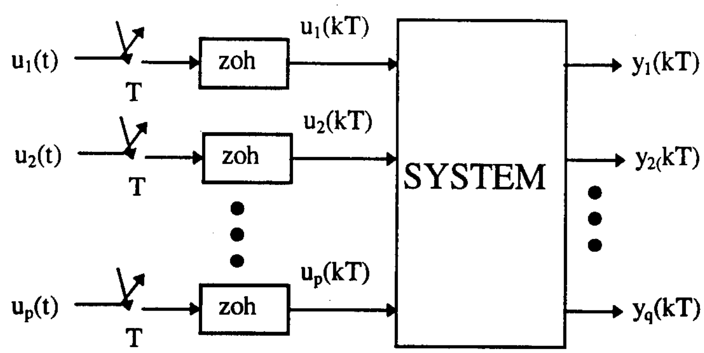

Figure 2-2 Linear Time-Varying Multivariable Discrete System

The resulting discrete system can be modeled by the following dynamic equations,

$$x((k+1)T) = F(k,T)x(kT) + G(k,T)u(kT) \tag{2-57}$$

and

$$y(kT) = C(kT)x(kT) + E(kT)u(kT), \tag{2-58}$$

where x(kT) is the (nx1) state vector, y(kT) is the (qx1) output vector, u(kT) is the (px1) input vector, F(k,T) is the (nxn) state matrix, G(k,T) is the (nxp) input matrix, C(kT) is the (qxn) output matrix, E(kT) is the (qxp) direct transmission matrix, and T is the sampling period. The appearance of the variable k in the matrices F(k,T), G(k,T), C(kT), and E(kT) implies that these matrices are time-varying. If the variable k does not appear in these matrices, then the system is assumed to be time-invariant. For the time-invariant case, the state equations simplify to

$$x((k+1)T) = F(T)x(kT) + G(T)u(kT) \tag{2-59}$$

and

$$y(kT) = C(T)x(kT) + E(T)u(kT).$$

(2-60)

2.3.1. Controllability

The concept of controllability plays an important role in the design of linear discrete systems. Controllability for this class of systems can be defined as follows:

Definition 2.5 *The linear discrete system given by* (2-57) *is said to be completely state controllable if there exists a piecewise-constant control vector u(kT) defined over a finite number of sampling periods such that, starting from any initial state, the state x(kT) can be transferred to any desired state* x_f *in at most n sampling periods.*

As is the case with linear continuous systems, a controllability matrix is used to test for the condition of complete state controllability. The controllability matrix S(T) of the linear time-invariant discrete system given in (2-59) is defined as

$$S(T) = \left[G(T) \quad F(T)G(T) \quad \cdots \quad F^{n-1}(T)G(T)\right].$$

(2-61)

The following theorem is used to test for the condition of complete state controllability for the class of linear time-invariant discrete systems [23]:

Theorem 2.1 *The linear time-invariant discrete system given by (2-59) is completely state controllable, if*

$$\mathrm{rank}\left[S(T)\right] = n.$$

(2-62)

A controllability theorem and the associated controllability matrix have yet to be developed for the class of linear time-varying discrete systems. Chapter 4 of this report will address the concept of controllability for this class of systems.

2.3.2. Equivalence Transformations

The use of equivalence transformations is an important technique in the design of linear time-invariant discrete control systems. Several techniques exist to transform such systems into particular canonical forms. The controllable canonical, observable canonical, and Jordan canonical forms are the most widely used methods available [25]. Such methods do not currently exist for the class of linear time-varying discrete systems. Chapter 5 of this report will address equivalence transformations for this class of systems.

2.3.3. Stabilization of Linear Discrete Systems

The concept of stability is important in the design of linear time-invariant discrete systems. Several stabilization techniques exist to allow the control system designer to develop a closed-loop control system in which the poles can be arbitrarily assigned to values within the unit circle of the complex plane. Ackermann's formula [25] is one such method. Such methods do not currently exist for the class of linear time-varying discrete systems. Chapter 5 of this report will also address the concept of stabilization for this class of systems.

2.4. Conclusion

This chapter presented relevant background information in the area of linear systems theory. Section 2.2 reviewed the research underlying the development of linear continuous systems. Section 2.3 presented relevant background information on linear discrete systems.

The concepts of controllability, canonical transformations, and stability were presented in-depth for the classes of linear time-invariant continuous and linear time-varying continuous systems. A limited discussion of these concepts was also provided for the linear time-invariant discrete class of systems. The remainder of this report focuses on developing these concepts for the class of linear time-varying discrete systems.

CHAPTER 3
DISCRETIZATION OF LINEAR TIME-VARYING CONTINUOUS SYSTEMS

3.1. Introduction

Most modern control systems consist of both continuous and discrete components. The typical system utilizes continuous signals as inputs to a microprocessor-based control system. These continuous signals must be discretized (converted into discrete signals) to be useful in the digital world of the microprocessor. Discretization is typically accomplished through the application of a sample and zero-order hold device to each continuous input. In order to model a discretized system, equations that define the relationship between the given continuous and the resulting discrete system must be developed.

3.2. Problem Statement

Equations must be developed which model the discretization of a linear time-varying multivariable continuous system. The following statement defines the specific problem to be addressed in this chapter:

Given a discretized system which results from the application of a sample and zero-order hold device to each input of a linear time-varying multivariable continuous system, derive equations to model the resulting discretized state equations (2-57) and (2-58) in terms of the associated continuous state equations (2-2) and (2-3).

The equations that model the resulting discretized system are derived in section 3.3. These results are illustrated through the use of examples in section 3.4. (Note: A second solution to the above problem can also be found in Ogata [25].)

3.3. Derivation of the Discretization Equations

The derivation of the discretization equations begins with the solution to the time-varying continuous system, given in [24] as

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau . \tag{3-1}$$

Since the inputs u(t) are the outputs of the zero-order hold devices,

$$u(t) = u(kT) \tag{3-2}$$

for $kT \le t \le (k+1)T$. Note that the initial condition at the beginning of the interval $[kT,(k+1)T]$ is

$$x_0 = x(t_0) = x(kT) . \tag{3-3}$$

To determine x(t) at the end of the interval $[kT,(k+1)T]$, substitute (3-2),(3-3),

$$t_0 = kT, \tag{3-4}$$

and

$$t = (k+1)T, \tag{3-5}$$

into (3-1). This yields

$$x((k+1)T) = \Phi((k+1)T, kT)x(kT) + \int_{kT}^{(k+1)T} \Phi((k+1)T, \tau)B(\tau)u(\tau)d\tau .$$

(3-6)

Since the input is constant between any two consecutive sampling periods, the input vector has the following property

$$u(\tau) = u(kT),$$

(3-7)

for kT≤τ≤(k+1)T. Substituting (3-7) into (3-6) yields

$$x((k+1)T) = \Phi((k+1)T, kT)x(kT) + \left[\int_{kT}^{(k+1)T} \Phi((k+1)T, \tau)B(\tau)d\tau\right]u(kT) .$$

(3-8)

A comparison of (3-8) and (2-57) shows that the discrete state matrix can be defined as

$$F(k, T) = \Phi((k+1)T, kT)$$

(3-9)

and the discrete input matrix can be defined as

$$G(k, T) = \int_{kT}^{(k+1)T} \Phi((k+1)T, \tau)B(\tau)d\tau .$$

(3-10)

The discretized output equation (2-58) is derived by substituting t=(kT) into the continuous output equation (2-3). The resulting output and direct transmission matrices are defined as

$$C(kT) = [C(t)]_{t=kT}$$

(3-11)

and

$$E(kT) = \left[E(t)\right]_{t=kT}.$$

(3-12)

Equations (3-9)-(3-12) are referred to as the discretization equations. These equations define the resulting linear time-varying discrete system after the application of sample and zero-order hold devices to the inputs of a linear time-varying multivariable continuous system. It should be noted that the discretization equations, (3-9) and (3-10), are dependent on the state transition matrix of the continuous system. In other words, the resulting discrete system can only be modeled if the state transition matrix of the continuous system can be computed. It can also be shown that the discretization equations derived here are equivalent to those in [25].

3.4. Examples

The following examples are used to illustrate the concepts of lexicografixed controllability and the discretization of linear time-varying continuous systems. Each example begins with a given linear time-varying continuous system. The property of lexicografixed controllability is then tested. If the test shows that the given system is not lexicografixed controllability then a sample and zero-order hold is applied to each input and the dynamic equations of the resulting discrete system are computed using (3-9)-(3-12). A step-by-step approach delineates the procedure necessary to discretize a given system. The Maple code used to solve these examples is included in Appendix A.

3.4.1. Example 3-1:

Consider a linear time-varying single-input-single-output continuous system with dynamic equations given as

$$\dot{x}(t) = \begin{bmatrix} -1 & \exp(-2t) \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ \exp(1-t) \end{bmatrix} u(t) \tag{3-13}$$

and

$$y(t) = \begin{bmatrix} \exp(-2t) & -1 \end{bmatrix} x(t) + u(t) . \tag{3-14}$$

a) **Check to see if the given system is lexicografixedly controllable.** From (2-18), the controllability matrix can be computed as

$$M(t) = \begin{bmatrix} M_1(t) & M_2(t) \end{bmatrix}, \tag{3-15}$$

where

$$M_1(t) = B(t) = \begin{bmatrix} 1 \\ \exp(1-t) \end{bmatrix} \tag{3-16}$$

and

$$M_2(t) = -A(t)B(t) + \frac{d}{dt}B(t) = \begin{bmatrix} 1-\exp(1-3t) \\ 0 \end{bmatrix}. \tag{3-17}$$

Substituting (3-16) and (3-17) into (3-15) yields

$$M(t) = \begin{bmatrix} 1 & 1-\exp(1-3t) \\ \exp(1-t) & 0 \end{bmatrix}.$$

(3-18)

Computing the determinant of (3-18) yields,

$$M_{det}(t) = \exp(2-4t) - \exp(1-t).$$

(3-19)

Equation (3-19) is plotted below.

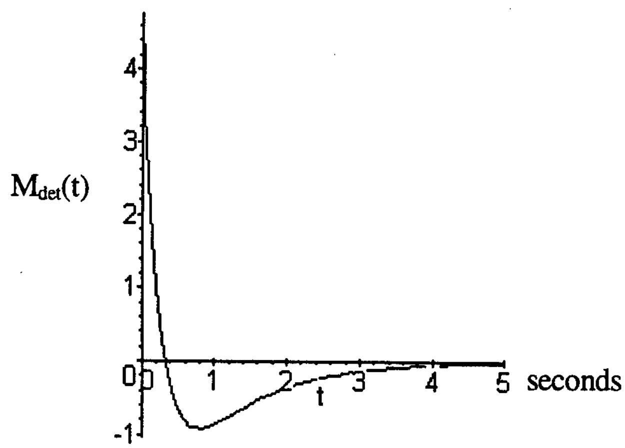

Figure 3-1 Determinant as a function of time

Figure 3-1 shows that the determinant of M(t) is zero for t=0.333 seconds. Therefore, from definition 2.3, the system is not lexicografixedly controllable.

b) **Check to see if the state matrix satisfies the commutative property.** Solving (2-8) yields

$$A(t)\left[\int_{t_0}^{t} A(\tau)d\tau\right] = \left[\int_{t_0}^{t} A(\tau)d\tau\right]A(t)$$
$$= \begin{bmatrix} t - t_0 & \left(\frac{1}{2} - t + t_0\right)\exp(-2t) - \frac{1}{2}\exp(-2t_0) \\ 0 & t - t_0 \end{bmatrix}.$$

(3-20)

Based on (3-20), it is obvious that the state matrix satisfies the commutative property.

c) **Compute the state transition matrix.** Since A(t) satisfies the commutative property, the state transition matrix is computed directly from (2-9).

$$\Phi(t,t_0) = \exp\left[\int_{t_0}^{t} A(\tau)d\tau\right] = \exp\begin{bmatrix} -t + t_0 & \frac{1}{2}\exp(2t) - \frac{1}{2}\exp(2t_0) \\ 0 & -t + t_0 \end{bmatrix}$$
$$= \begin{bmatrix} \exp(-t + t_0) & \frac{1}{2}\exp(-t - t_0) - \frac{1}{2}\exp(-3t + t_0) \\ 0 & \exp(-t + t_0) \end{bmatrix}$$

(3-21)

d) **Assume that a sample and zero-order hold is applied to the input of the continuous system. Use the discretization equations to model the resulting discrete system.** From (3-9), the discretized state matrix is computed as

$$F(k,T) = \Phi((k+1)T,kT) = \begin{bmatrix} \exp(-T) & \frac{1}{2}\exp(-(1+2k)T) - \frac{1}{2}\exp(-(3+2k)T) \\ 0 & \exp(-T) \end{bmatrix},$$

(3-22)

and from (3-10) the discretized input matrix is computed as

$$G(k,T) = \int_{kT}^{(k+1)T} \Phi((k+1)T,\tau)B(\tau)d\tau$$

$$= \begin{bmatrix} \left(-\frac{1}{4}-\frac{1}{2}T\right)\exp(1-3kT-3T)+1-\exp(-T)+\frac{1}{4}\exp(1-3kT-T) \\ T\exp(1-kT-T) \end{bmatrix}.$$

(3-23)

From (3-11) and (3-12) the discrete output and direct transmission matrices are computed as

$$C(kT) = \begin{bmatrix} \exp(-2kT) & -1 \end{bmatrix}$$

(3-24)

and

$$E(kT) = 1.$$

(3-25)

The dynamic equations of the resulting system can be obtained by substituting (3-22)-(3-25) into (2-57) and (2-58).

3.4.2. Example 3-2:

Consider a linear time-varying multivariable continuous system with dynamic equations given as

$$\dot{x}(t) = \begin{bmatrix} \exp(-t) & 0 \\ 0 & \exp(-t) \end{bmatrix} x(t) + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} u(t)$$

(3-26)

and

$$y(t) = \begin{bmatrix} t & 1 \end{bmatrix} x(t) + \begin{bmatrix} t & 1 \end{bmatrix} u(t).$$

(3-27)

a) **Check to see if the given system is lexicografixedly controllable.** From (2-18), the controllability matrix can be computed as

$$M(t) = \begin{bmatrix} M_1(t) & M_2(t) \end{bmatrix}, \tag{3-28}$$

where

$$M_1(t) = B(t) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{3-29}$$

and

$$\begin{aligned} M_2(t) &= -A(t)B(t) + \frac{d}{dt}B(t) = -\begin{bmatrix} \exp(-t) & 0 \\ 0 & \exp(-t) \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{d}{dt}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -\exp(-t) & -\exp(-t) \\ -\exp(-t) & -\exp(-t) \end{bmatrix}. \end{aligned} \tag{3-30}$$

Substituting (3-29) and (3-30) into (3-28) yields

$$M(t) = \begin{bmatrix} 1 & 1 & -\exp(-t) & -\exp(-t) \\ 1 & 1 & -\exp(-t) & -\exp(-t) \end{bmatrix}. \tag{3-31}$$

From (3-31) it is obvious that the determinant of M(t) is zero for all t. Therefore M(t) does not have full rank and the given system is not lexicografixedly controllable.

b) **Check to see if the state matrix satisfies the commutative property of (2-8).** Since A(t) is a diagonal matrix, it satisfies the commutative property [6].

c) **Compute the state transition matrix.** Since A(t) satisfies the commutative property, the state transition matrix is computed directly from (2-9).

$$\Phi(t,t_0) = \exp\left[\int_{t_0}^{t} A(\tau)d\tau\right] = \begin{bmatrix} \exp\left(-\exp(-t)+\exp(-t_0)\right) & 0 \\ 0 & \exp\left(-\exp(-t)+\exp(-t_0)\right) \end{bmatrix}$$

(3-32)

d) **Assume that a sample and zero-order hold is applied to each input of the continuous system. Use the discretization equations to model the resulting discrete system.** From (3-9), the discretized state matrix is computed as

$$\begin{aligned} F(k,T) &= \Phi((k+1)T,kT) \\ &= \begin{bmatrix} \exp\left(-\exp\left(-(k+1)T\right)+\exp(-kT)\right) & 0 \\ 0 & \exp\left(-\exp\left(-(k+1)T\right)+\exp(-kT)\right) \end{bmatrix} \end{aligned}$$

(3-33)

and from (3-10) the discretized input matrix is computed as

$$G(k,T) = \int_{kT}^{(k+1)T} \Phi((k+1)T,\tau)B(\tau)d\tau = \begin{bmatrix} \alpha(k,T) & \alpha(k,T) \\ \alpha(k,T) & \alpha(k,T) \end{bmatrix},$$

(3-34)

where

$$\alpha(k,T) = \frac{Ei(1,-\exp(-(k+1)T)) - Ei(1,-\exp(-kT))}{\exp(\exp(-(k+1)T))}$$

(3-35)

and the function Ei(n,x) is defined in [14] as

$$Ei(n,x) = \int_{1}^{\infty} \frac{\exp\left(-x^{t}\right)}{t^{n}} dt \quad \text{for } Re(x) > 0.$$

(3-36)

From (3-11) and (3-12) the discrete output and direct transmission matrices are computed as

$$C(kT) = \begin{bmatrix} kT & 1 \end{bmatrix}$$

(3-37)

and

$$E(kT) = \begin{bmatrix} kT & 1 \end{bmatrix}.$$

(3-38)

The dynamic equations of the resulting system can be obtained by substituting (3-33)-(3-38) into (2-57) and (2-58).

3.4.3. Example 3-3:

Consider a linear time-varying multivariable continuous system with state equation given by

$$\dot{x}(t) = \begin{bmatrix} 2-4\exp(-2t) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{3t+1}{2(t+1)} \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} u(t).$$

(3-39)

a) **Check to see if the given system is lexicografixedly controllable.** From (2-18), the controllability matrix can be computed as

$$M(t) = \begin{bmatrix} M_1(t) & M_2(t) & M_3(t) \end{bmatrix},$$

(3-40)

where

$$M_1(t) = B(t) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix},$$

(3-41)

$$M_2(t) = -A(t)B(t) + \frac{d}{dt}B(t) = \begin{bmatrix} -2+4\exp(-2t) & 0 \\ -1 & -1 \\ 0 & \dfrac{-(3t-1)}{2(t+1)} \end{bmatrix},$$

(3-42)

and

$$M_3(t) = -A(t)Q_2(t) + \frac{d}{dt}Q_2(t) = \begin{bmatrix} \begin{pmatrix} 4-24\exp(-2t) \\ +16\exp(-4t) \end{pmatrix} & 0 \\ 1 & 1 \\ 0 & \left(\dfrac{9t^2-6t-7}{4(t+1)^2}\right) \end{bmatrix}.$$

(3-43)

Substituting (3-41), (3-42), and (3-43) into (3-40) yields

$$M(t) = \begin{bmatrix} 1 & 0 & -2+4\exp(-2t) & 0 & \begin{matrix} 4-24\exp(-2t) \\ +16\exp(-4t) \end{matrix} & 0 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 0 & 1 & 0 & \dfrac{\frac{-3}{2}t+\frac{1}{2}}{t+1} & 0 & \dfrac{9t^2-6t-7}{4(t+1)^2} \end{bmatrix}.$$

(3-44)

This system is lexicografixedly controllable if the determinant of any matrix consisting of any combination of any three column vectors of M(t) is non-zero for all time t [21]. There are 20 matrices that can be formed which consist of three column vectors of M(t). Since the determinants of all 20 matrices were found to be non-zero for all t, the system is not lexicografixedly controllable in continuous time.

b) **Check to see if the state matrix satisfies the commutative property of (2-8).**

Since A(t) is a diagonal matrix, it satisfies the commutative property [6].

c) **Compute the state transition matrix.** Since A(t) satisfies the commutative property, the state transition matrix is computed directly from (2-9).

$$\Phi(t,t_0) = \exp\left[\int_{t_0}^{t} A(\tau)d\tau\right]$$

$$= \begin{bmatrix} \exp\begin{pmatrix} 2t + 2\exp(-2t) \\ -2t_0 - 2\exp(-2t_0) \end{pmatrix} & 0 & 0 \\ 0 & \exp(t - t_0) & 0 \\ 0 & 0 & \dfrac{(t_0+1)^2}{(t+1)^2}\exp\left(\dfrac{3}{2}t - \dfrac{3}{2}t_0\right) \end{bmatrix}$$

(3-45)

d) **Assume that a sample and zero-order hold is applied to each input of the continuous system. Use the discretization equations to model the resulting discrete system.** From (3-9), the discretized state matrix is computed as

$$
F(k,T) = \Phi((k+1)T, kT)
$$

$$
= \begin{bmatrix} \exp\begin{pmatrix} 2\exp(-2(k+1)T) \\ +2T - 2\exp(-2kT) \end{pmatrix} & 0 & 0 \\ 0 & \exp(T) & 0 \\ 0 & 0 & \frac{(kT+1)^2}{(kT+T+1)^2}\exp\left(\frac{3}{2}T\right) \end{bmatrix},
$$

(3-46)

and from (3-10) the discretized input matrix is computed as

$$
G(k,T) = \int_{kT}^{(k+1)T} \Phi((k+1)T, \tau)B(\tau)d\tau = \begin{bmatrix} g_{00}(T) & 0 \\ \exp(T)-1 & \exp(T)-1 \\ 0 & g_{21}(T) \end{bmatrix},
$$

(3-47)

where

$$
g_{00}(T) = \frac{1}{4}\exp(2(k+1)T) - \frac{1}{2}\exp(-\exp(-2kT) + kT + T + \exp(-2(k+1)T))
$$

(3-48)

and

$$
g_{21}(T) = \frac{2(30kT + 29 + 9k^2T^2)}{27(kT+T+1)^2}\exp\left(\frac{3}{2}T\right) + \frac{2(-30kT - 9T^2 - 29 - 9k^2T^2 - 18kT^2 - 30T)}{27(kT+T+1)^2}.
$$

(3-49)

The state equation of the resulting system can be obtained by substituting (3-46)-(3-47) into (2-57).

3.5. Conclusion

Linear time-varying continuous systems are discretized through the application of a sample and zero-order hold to each input of the system. Equations modeling the resulting discrete system are derived in section 3.3. These results show that the discrete model is dependent on the state transition matrix of the given continuous system. That is, if the state transition matrix of a continuous system can be computed, then a discretized system can be modeled through the application of the discretization equations derived in this chapter. The discretization process was successfully demonstrated in section 3.4 through the use of several examples.

CHAPTER 4
CONTROLLABILITY OF LINEAR TIME-VARYING DISCRETE SYSTEMS

4.1. Introduction

The property of controllability deals with the existence of a control vector that can cause the system's state to reach some arbitrary state [25] in a finite period of time. A given system must possess the property of controllability if state feedback and stabilization techniques are to be used to control the system. For linear continuous systems, these techniques rely heavily on the continuous controllability matrix (2-18) which is defined for both time-invariant and time-varying continuous systems. For discrete systems, these techniques rely heavily on the discrete controllability matrix (2-61) which is defined only for the time-invariant discrete class of systems. However, in the case of linear time-varying discrete systems, a controllability matrix has not yet been defined. Consequently, state feedback and stabilization techniques do not exist for this class of systems.

4.2. Problem Statement

In order to develop state feedback and stabilization techniques for linear time-varying discrete systems, a controllability theorem and its corresponding controllability matrix must be developed. The following statement defines the specific problem to be solved in this chapter:

Given definition 2.5 which describes the concept of complete state controllability for linear discrete systems, derive a controllability matrix and develop a controllability

theorem which can be used to test for the condition of complete state controllability.

A controllability matrix for this class of systems is derived and the associated controllability theorem is developed in section 4.3. These results are illustrated through the use of examples in section 4.4.

4.3. Controllability Matrix and Theorem for the Class of Linear Time-Varying Discrete Systems

In this section a new controllability theorem for linear time-varying multivariable discrete systems is introduced and its associated controllability matrix is derived. These results represent advances in the theory of the control of linear discrete systems.

Using definition 2.5 which defines complete state controllability for linear discrete systems, a necessary and sufficient condition for the complete state controllability of a linear time-varying multivariable discrete system is derived. The derivation begins by finding the solution to the discrete state equation (2-57).

The solution of (2-57) can be found by recursion, as follows:

For k=0,

$$x(T) = F(0,T)x(0) + G(0,T)u(0). \tag{4-1}$$

For k=1,

$$x(2T) = F(1,T)F(0,T)x(0) + F(1,T)G(0,T)u(0) + G(1,T)u(T). \tag{4-2}$$

For k=2,

$$x(3T) = F(2,T)F(1,T)F(0,T)x(0) + F(2,T)F(1,T)G(0,T)u(0) + F(2,T)G(1,T)u(T) + G(2,T)u(2T). \tag{4-3}$$

For k=3,

$$x(4T) = F(3,T)F(2,T)F(1,T)F(0,T)x(0) + F(3,T)F(2,T)F(1,T)G(0,T)u(0) + F(3,T)F(2,T)G(1,T)u(T) + F(3,T)G(2,T)u(2T) + G(3,T)u(3T). \tag{4-4}$$

Repeating this procedure, gives

$$x(kT) = [F(k-1,T)F(k-2,T)\dots F(0,T)]x(0) + \sum_{j=0}^{k-1} [[F(k-1,T)F(k-2,T)\dots F(j+1,T)]G(jT)u(jT). \tag{4-5}$$

The solution to the state equation (2-57) at the nth sampling period can be obtained by substituting k=n into (4-5). This yields

$$x(nT) = [F(n-1,T)F(n-2,T)\dots F(0,T)]x(0) + \sum_{j=0}^{n-1} [[F(n-1,T)F(n-2,T)\dots F(j+1,T)]G(jT)u(jT). \tag{4-6}$$

Rewriting (4-6) yields

$$\begin{aligned} x(nT) - [F(n-1,T)F(n-2,T)\dots F(0,T)]x(0) \\ &= \sum_{j=0}^{n-1} [F(n-1,T)F(n-2,T)\dots F(j+1,T)]G(j,T)u(jT) \\ &= [F(n-1,T)F(n-2,T)\dots F(1,T)]G(0,T)u(0) \\ &+[F(n-1,T)F(n-2,T)\dots F(2,T)]G(1,T)u(T) \\ &+[F(n-1,T)F(n-2,T)\dots F(3,T)]G(2,T)u(2T) \\ &+\dots+G(n-1,T)u((n-1)T) \end{aligned} \tag{4-7}$$

which can be written in matrix form as follows,

$$x(nT) - [F(n-1,T)F(n-2,T)\ldots F(0,T)]x(0)$$
$$= \begin{bmatrix} G(n-1,T) \\ F(n-1,T)G(n-2,T) \\ \vdots \\ [F(n-1,T)F(n-2,T)\ldots F(1,T)]G(0,T) \end{bmatrix}^T \begin{bmatrix} u((n-1)T) \\ u((n-2)T) \\ \vdots \\ u(0) \end{bmatrix}$$

(4-8)

The controllability matrix S(T) for the class of linear time-varying multivariable discrete systems is now defined as

$$S(T) = \begin{bmatrix} G(n-1,T) \\ F(n-1,T)G(n-2,T) \\ \vdots \\ [F(n-1,T)F(n-2,T)\ldots F(1,T)]G(0,T) \end{bmatrix}^T .$$

(4-9)

In compact form, the controllability matrix (4-9) can be written as

$$S(T) = [S_0(T) \quad S_1(T) \quad \ldots \quad S_{n-1}(T)],$$

(4-10)

where

$$S_0(T) = G(n-1,T),$$

(4-11)

and

$$S_i(T) = \left[\prod_{j=1}^{i} F(n-j,T) \right] G(n-1-i,T).$$

(4-12)

Since the input matrix G(k,T) is an (nxp) matrix, each of the matrices

$S_0(T), S_1(T), \ldots, S_{n-1}(T)$ of (4-10) is an (nxp) matrix. Therefore, the controllability matrix S(T) is an (nxnp) matrix. For time-invariant discrete systems, the time-varying controllability matrix (4-10) reduces to the time-invariant controllability matrix (2-61).

A new controllability theorem will now be introduced and subsequently proven.

Theorem 4.1

The state equation (2-57) of a linear time-varying multivariable discrete system is completely state controllable, if and only if the rank of the (nxnp) controllability matrix is n at some sampling period T.

To prove the necessity of theorem 4.1, assume that the state equation is state controllable and then show that the controllability matrix has full rank. The state transition equation of the discrete system can be obtained by substituting (4-9) into (4-8).

$$x(nT) - [F(n-1,T)F(n-2,T)\ldots F(0,T)]x(0) = S(T)\begin{bmatrix} u((n-1)T) \\ u((n-2)T) \\ \vdots \\ u(0) \end{bmatrix}. \tag{4-13}$$

If the left hand side of (4-13) is represented as an (nx1) vector X(nT) then (4-13) can be written as

$$X(nT) = S(T)U(nT). \tag{4-14}$$

From definition 2.5, if the system is assumed to be completely state controllable, then every initial state X(0) can be transferred by unconstrained controls U(nT) to any final state X(nT) in a finite number of sampling periods. Thus the problem can be reduced to the

following: Given S(T) and every vector X(nT) in the n-dimensional state space, solve for the controls U(nT). Since (4-14) represents n simultaneous linear equations, these equations must be linearly independent for solutions to exist. The controllability matrix S(T) must, therefore, have full rank (equal to n) for solutions to exist. Necessity has been proven.

Contradiction is used to prove the sufficiency of the theorem. Assume that the controllability matrix does not have full rank and that the system is completely state controllable. If the controllability matrix does not have full rank, then, from the theory of linear equations, S(T) does not have n linearly independent column vectors. If S(T) does not have at least n linearly independent column vectors, then, given X(nT) and U(nT), the set of simultaneous linear equations (4-14) cannot be solved. This means that unconstrained controls capable of transferring every initial state X(0) to any final state X(nT) cannot be found. This contradicts the assumption that the system is completely state controllable. Sufficiency has been proven. Consequently, the full rank condition given in theorem 4.1 is found to be both a necessary and sufficient condition for complete state controllability. The proof is complete.

4.4. Examples

The following examples are used to illustrate Theorem 4.1 and its corresponding controllability matrix. The Maple code used to solve these examples is included in Appendix A.

4.4.1. Example 4-1:

Consider a linear time-varying single-input-single-output discrete system with state and input matrices given by (3-22) and (3-23).

a) **Compute the controllability matrix of the given system.** From (4-10), the controllability matrix can be computed as

$$S(T)=\begin{bmatrix}\begin{pmatrix}1-\left(\frac{1}{4}+\frac{T}{2}\right)\exp(1-6T)\\-\exp(-T)+\frac{\exp(1-4T)}{4}\end{pmatrix} & \begin{pmatrix}\exp(-T)-\frac{\exp(1-4T)}{4}-\exp(-2T)\\+\frac{\exp(1-2T)}{4}-\frac{T\exp(1-6T)}{2}\end{pmatrix}\\ T\exp(1-2T) & T\exp(1-2T)\end{bmatrix}.$$

(4-15)

b) **Determine whether the given system is completely state controllable.** Computing the determinant of the controllability matrix (4-15) yields,

$$\begin{aligned}S_{det}(T) &= T\exp(1-2T)+T\exp(1-4T)-\frac{T\exp(2-4T)}{4}\\ &-\frac{T\exp(2-8T)}{4}-2T\exp(1-3T)+\frac{T\exp(2-6T)}{2}.\end{aligned}$$

(4-16)

Theorem 4.1 states that a given discrete system is completely state controllable if and only if the controllability matrix has full rank. It is well known that a matrix which is dependent on the variable T has full rank at values of T for which the determinant is non-zero. Plotting (4-16) as a function of the sampling period T yields the graph depicted in Figure 4-1.

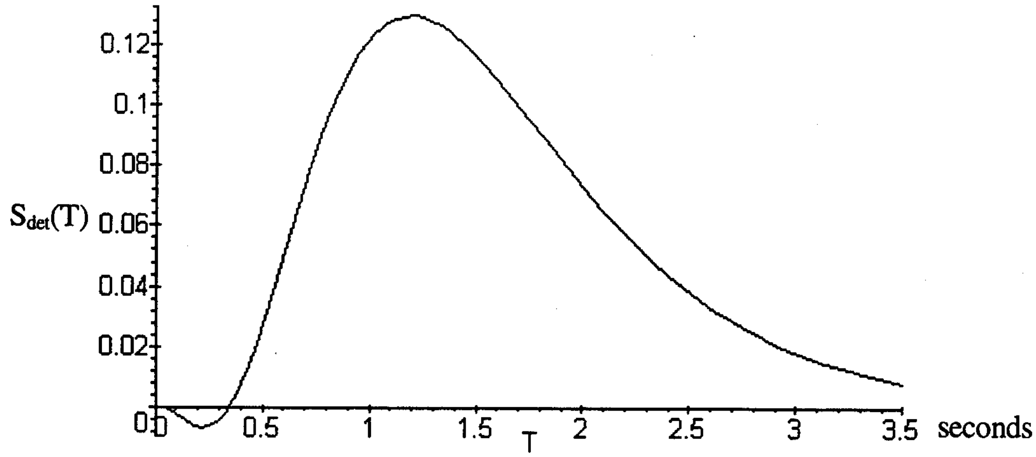

Figure 4-1 Determinant as a Function of Sampling Period (Example 4-1)

Figure 4-1 shows that the value of $S_{det}(T)$ crosses the zero axis (i.e., equals exactly zero) at two values of T and also approaches zero as T increases. Solving (4-16) shows that $S_{det}(T)=0$, and the system is therefore uncontrollable, for sampling periods of T=0 and T=0.343 seconds. Theoretically, the system is controllable at any other sampling period. However, choosing a sampling period at which $S_{det}(T)$ is very close to zero is undesirable because the inverse of $S_{det}(T)$ would then approach infinity and the system would be ill-defined. For practical purposes it is desirable to select a sampling period such that $S_{det}(T) > |10^{-3}|$. Using this criteria, the range of acceptable sampling periods can be computed as

$$
\begin{aligned}
&0.08 < T < 0.33 \text{ seconds} \\
&0.35 < T < 4.72 \text{ seconds.}
\end{aligned}
\tag{4-17}
$$

c) **Demonstrate that the given system is completely state controllable.** Assume a sampling period of T=0.5 seconds, an initial state at k=0 of

$$\begin{bmatrix} x_0(0) \\ x_1(0) \end{bmatrix} = \begin{bmatrix} 2.0 \\ 5.0 \end{bmatrix},$$

(4-18)

and a desired state at k=n=2 of

$$\begin{bmatrix} x_0(2) \\ x_1(2) \end{bmatrix} = \begin{bmatrix} 0.5 \\ 2.5 \end{bmatrix}.$$

(4-19)

At a sampling period of T=0.5 seconds, the state equation becomes

$$\begin{bmatrix} x_o(k+1) \\ x_1(k+1) \end{bmatrix} = \begin{bmatrix} 0.6065306597 & 0.1917002498\exp(-k) \\ 0 & 0.6065306597 \end{bmatrix} \begin{bmatrix} x_0(k) \\ x_1(k) \end{bmatrix} + \begin{bmatrix} 0.108914879\exp(-1.5k) + 0.3934693403 \\ 0.8243606355\exp(-0.5k) \end{bmatrix} u(k).$$

(4-20)

Substituting k=0 into (4-20) yields,

$$\begin{bmatrix} x_0(1) \\ x_1(1) \end{bmatrix} = \begin{bmatrix} 2.171562568 + 0.5023843282\,u(0) \\ 3.032653299 + 0.8243606355\,u(0) \end{bmatrix}.$$

(4-21)

Substituting k=1 into (4-20) yields

$$\begin{bmatrix} x_0(2) \\ x_1(2) \end{bmatrix} = \begin{bmatrix} 1.530989814 + 0.3628475375\,u(0) + 0.4177715590\,u(1) \\ 1.839397206 + 0.5000000001\,u(0) + 0.5000000001\,u(1) \end{bmatrix}.$$

(4-22)

By substituting (4-19) into (4-22) and then solving the resulting set of simultaneous linear equations, it can be shown that the control vector

$$\begin{bmatrix} u(0) \\ u(1) \end{bmatrix} = \begin{bmatrix} 28.82075800 \\ -27.49955241 \end{bmatrix}$$

(4-23)

will produce the desired state (4-19). Hence, by definition 2.5 the given discrete system is completely state controllable.

4.4.2. Example 4-2:

Consider a linear time-varying multiple-input discrete system with state and input matrices given by (3-33) and (3-34).

a) **Compute the controllability matrix of the given system.** From (4-10), the controllability matrix can be computed as

$$S(T) = \begin{bmatrix} \alpha(T) & \alpha(T) & \beta(T) & \beta(T) \\ \alpha(T) & \alpha(T) & \beta(T) & \beta(T) \end{bmatrix},$$

(4-24)

where

$$\alpha(T) = \frac{Ei(1,-exp(-2T)) - Ei(1,-exp(-T))}{exp(exp(-2T)}$$

(4-25)

and

$$\beta(T) = \left(Ei\left(1,-exp(-T)\right) - Ei\left(1,-1\right)\right)exp\left(-exp(-T)\right)exp\left(-exp(-2T)+exp(-T)\right).$$

(4-26)

b) **Determine whether the given system is completely state controllable.** By inspection of (4-24), it is obvious that the determinant of the controllability matrix equals zero for all sampling periods. Hence, the discrete system is not completely state controllable.

4.4.3. Example 4-3:

Consider a linear time-varying multiple-input discrete system with state and input matrices given by (3-46) and (3-47).

a) **Compute the controllability matrix of the given system.** From (4-10), the controllability matrix can be computed as

$$S(T)=\begin{bmatrix} S0_{0,0} & S0_{0,1} & S1_{0,0} & S1_{0,1} & S2_{0,0} & S2_{0,1} \\ S0_{1,0} & S0_{1,1} & S1_{1,0} & S1_{1,1} & S2_{1,0} & S2_{1,1} \\ S0_{2,0} & S0_{2,1} & S1_{2,0} & S1_{2,1} & S2_{2,0} & S2_{2,1} \end{bmatrix},$$

(4-27)

where the matrix elements of S(T) are given as

$$S0_{0,0}=\frac{1}{4}\exp(6T)-\frac{1}{4}\exp(-2\exp(-4T)+6T+2\exp(-6T)),$$

(4-28)

$$S0_{1,0}=S0_{1,1}=\exp(T)-1,$$

(4-29)

$$S0_{2,0}=S0_{0,1}=S1_{0,1}=S1_{2,0}=S2_{2,0}=S2_{0,1}=0,$$

(4-30)

$$S0_{2,1} = \frac{\left(72T^2 + 120T + 58\right)\exp\left(\frac{3T}{2}\right) - 2\left(81T^2 + 90T + 29\right)}{27(3T+1)^2},$$

(4-31)

$$S1_{0,0} = \frac{1}{4}\exp\begin{pmatrix}2T + 2\exp(-6T) \\ -2\exp(-4T)\end{pmatrix}\left(\exp(4T) - \exp\begin{pmatrix}-2\exp(-2T) + 4T \\ +2\exp(-4T)\end{pmatrix}\right),$$

(4-32)

$$S1_{1,0} = S1_{1,1} = \exp(T)(-1 + \exp(T)),$$

(4-33)

$$S1_{2,1} = \frac{2\exp\left(\frac{3}{2}T\right)}{27(3T+1)^2}\left(\left(9T^2 + 30T + 29\right)\exp\left(\frac{3}{2}T\right) - \left(36T^2 + 60T + 29\right)\right),$$

(4-34)

$$S2_{0,0} = \frac{1}{4}\left(\exp(2T) - \exp\left(-2 + 2T + 2\exp(-2T)\right)\right)\exp\left(4T + 2\exp(-6T) - 2\exp(-2T)\right),$$

(4-35)

$$S2_{1,0} = S2_{1,1} = \exp(2T)\left(-1 + \exp(T)\right),$$

(4-36)

and

$$S2_{2,1} = \frac{-2\exp(3T)}{27(3T+1)^2}\left(9T^2 + 30T + 29 - 29\exp\left(\frac{3}{2}T\right)\right).$$

(4-37)

(b) **Determine whether the given system is completely state controllable.** Theorem 4.1 states that a given discrete system is completely state controllable if and only if the

controllability matrix has full rank. Since (4-27) is a (3x6) matrix, the controllability matrix has full rank at values of T for which the determinant of any (3x3) sub-matrix is non- zero. A (3x3) sub-matrix is formed by taking the first three column vectors of (4-27).

$$S_{sub}(T) = \begin{bmatrix} S0_{0,0} & S0_{0,1} & S1_{0,0} \\ S0_{1,0} & S0_{1,1} & S1_{1,0} \\ S0_{2,0} & S0_{2,1} & S1_{2,0} \end{bmatrix}$$

(4-38)

Computing the determinant of (4-38) yields,

$$S_{det}(T) = \frac{(-1+\exp(T))}{54(3T+1)^2}\left(-81T^2 - 90T - 29 + (36T^2 + 60T + 29)\exp\left(\frac{3}{2}T\right)\right)$$
$$\begin{pmatrix} -\exp(7T) + \exp(7T - 2\exp(4T) + 2\exp(6T)) \\ +\exp(-2\exp(4T) + 6T + 2\exp(6T)) \\ -\exp(6T + 2\exp(-6T) - 2\exp(-2T)) \end{pmatrix}.$$

(4-39)

Plotting (4-39) as a function of the sampling period T yields the graph depicted in Figure 4-2.

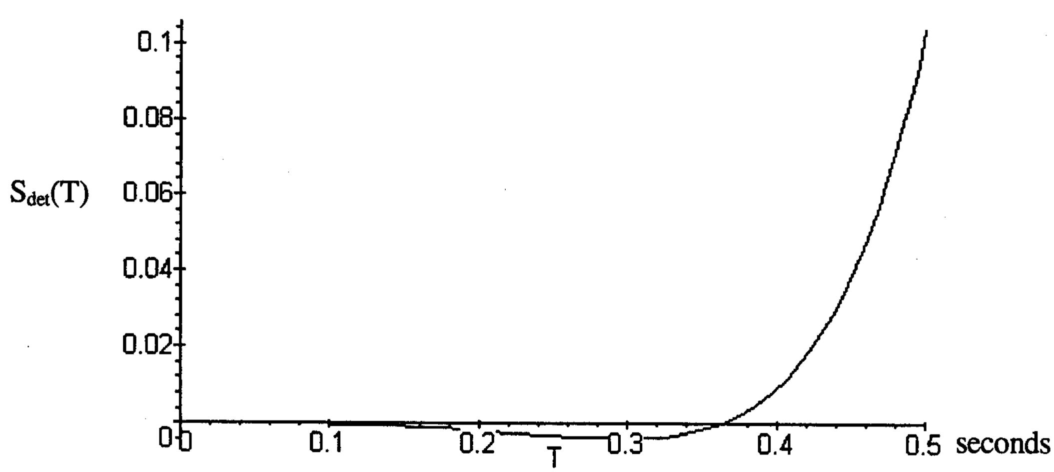

Figure 4-2 Determinant as a Function of Sampling Period (Example 4-3)

Figure 4-2 shows that the value of $S_{det}(T)$ crosses the zero axis (i.e., equals exactly zero) at two values of T and approaches infinity as T increases. Solving (4-39) shows that $S_{det}(T)=0$, and the system is therefore uncontrollable, at sampling periods of T=0 and T=0.363 seconds. Theoretically, the system is controllable at any other sampling period. However, choosing a sampling period at which $S_{det}(T)$ is very close to zero or infinity is undesirable because the inverse of $S_{det}(T)$ would then approach infinity or zero, respectively, and the system would be ill-defined. For practical purposes it is desirable to select a sampling period such that $\left|10^{-3}\right| < S_{det}(T) < \left|10^{6}\right|$. Using this criteria, the range of acceptable sampling periods can be computed as

$$\begin{gathered} 0.17 < T < 0.35 \text{ seconds} \\ 0.37 < T < 2.40 \text{ seconds.} \end{gathered} \tag{4-40}$$

c) **Demonstrate that the given system is completely state controllable.** Assume a sampling period of T=0.5 seconds, an initial state at k=0 of

$$\begin{bmatrix} x_0(0) \\ x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2.0 \\ 5.0 \\ 1.0 \end{bmatrix}, \tag{4-41}$$

and a desired state at k=n=3 of

$$\begin{bmatrix} x_0(3) \\ x_1\ (3) \\ x_2\ (3) \end{bmatrix} = \begin{bmatrix} 0.5 \\ 2.5 \\ 0 \end{bmatrix}.$$

(4-42)

At a sampling period of T=0.5 seconds, the state equation becomes

$$x(k+1) = F(k)\,x(k) + G(k)\,u(k)\,,$$

(4-43)

where

$$F(k) = \begin{bmatrix} \exp(1+2\exp(-k-1)-2\exp(-k)) & 0 & 0 \\ 0 & 1.648721271 & 0 \\ 0 & 0 & 2.117000017\dfrac{(k+2)^2}{(k+3)^2} \end{bmatrix}$$

(4-44)

and

$$G(k) = \begin{bmatrix} \begin{pmatrix} 0.6795704570\exp(k) \\ (1-\exp(-1.264241118\exp(-k))) \end{pmatrix} & 0 \\ 0.648721271 & 0.648721271 \\ 0 & \left(\dfrac{\begin{pmatrix} 0.1861666695k^2 + 0.907777797k \\ +1.12170374 \end{pmatrix}}{0.25k^2 + 1.5k + 2.25}\right) \end{bmatrix}.$$

(4-45)

Substituting k=0 into (4-43) yields,

$$\begin{bmatrix} x_0(1) \\ x_1(1) \\ x_2(1) \end{bmatrix} = \begin{bmatrix} 1.535576779 + 0.4876233596u_0(0) \\ 8.243606355 + 0.648721271(u_0(0) + u_1(0)) \\ 0.9408888964 + 0.4985349956u_1(0) \end{bmatrix}.$$

(4-46)

Substituting k=1 into (4-43) yields,

$$\begin{bmatrix} x_0(2) \\ x_1(2) \\ x_2(2) \end{bmatrix} = \begin{bmatrix} 2.621686325 + 0.8325181203u_0(0) + 0.687035113u_0(1) \\ 13.59140915 + 1.069560558(u_0(0) + u_1(0)) + 0.648721271(u_0(1) + u_1(1)) \\ 1.120422268 + 0.5936617094u_1(0) + 0.5539120516u_1(1) \end{bmatrix}.$$

(4-47)

Substituting k=2 into (4-43) yields,

$$\begin{bmatrix} x_0(3) \\ x_1(3) \\ x_2(3) \end{bmatrix} = \begin{bmatrix} 6.005773861 + 1.907137218u_0(0) + 1.573863922u_0(1) + 0.789661357u_0(2) \\ \begin{pmatrix} 22.40844537 + 1.763407243(u_0(0) + u_1(0)) \\ +1.069560558(u_0(1) + u_1(1)) + 0.648721271(u_0(2) + u_1(2)) \end{pmatrix} \\ 1.518037735 + 0.8043403834u_1(0) + 0.7504843666u_1(1) + 0.589108162u_1(2) \end{bmatrix}.$$

(4-48)

By substituting (4-42) into (4-48) and then solving the resulting set of simultaneous linear equations, it can be shown that the following set of control inputs

$$\begin{Bmatrix} u_0(1) = u_0(1), u_1(1) = u_1(1), u_0(2) = u_0(2), \\ u_0(0) = -2.88693116 - 0.8252494404u_0(1) - 0.4140558684u_0(2), \\ u_1(0) = -14.97818061 + 0.4394459752u_0(1) + 0.09277687566u_0(2) - 0.2770198232u_1(1), \\ u_1(2) = 17.87365798 - 0.5999987219u_0(1) - 0.126673152u_0(2) - 0.8957033186u_1(1) \end{Bmatrix}$$

(4-49)

will produce the desired state (4-42). Since the set of simultaneous linear equations contains three equations and six unknowns, the solution set (4-49) is not unique. The control system designer can assign any value to $u_0(1)$, $u_1(1)$, and $u_0(2)$. The required values for $u_0(0)$, $u_1(0)$,

and $u_1(2)$ can then be computed by substituting the selected values of $u_0(1)$, $u_1(1)$, and $u_0(2)$ into (4-49) and solving the resulting set of equations. For example, if the following set of inputs is assigned,

$$\{u_0(1) = 2.5,\ u_0(2) = 0, u_1(1) = 2.5\},$$

(4-50)

then it can be shown that inputs of

$$\{u_1(2) = 14.134,\ u_1(0) = -14.572,\ u_0(0) = -4.950\}$$

(4-51)

will produce the desired state (4-42). Hence, by definition 2.5 the given discrete system is completely state controllable.

4.5. Conclusion

Until now, the concept of controllability had not been applied to linear time-varying multivariable discrete systems. The results of the original research on the controllability of this class of systems are presented in this chapter. A new controllability theorem is developed and its associated controllability matrix is derived in section 4.3. A proof demonstrating the full rank condition of the controllability matrix as a necessary and sufficient condition for the system to be completely state controllable is provided. The validity of the controllability matrix and theorem is demonstrated in section 4.4 through the use of several examples. These examples successfully demonstrate that the state variables of a linear time-varying discrete system which is completely state controllable can be transferred

to any desired state in a finite number of sampling periods. The controllability theorem and matrix developed in this chapter represent advances in the theory of linear system control.

CHAPTER 5
STABILIZATION OF LINEAR TIME-VARYING DISCRETE SYSTEMS

5.1. Introduction

State feedback techniques and controllability concepts play important roles in the stabilization and optimization of linear systems. If the state equation of a linear system is controllable, then all its eigenvalues can be arbitrarily assigned through the use of state feedback [6]. This flexibility in eigenvalue assignment allows the control system designer to stabilize the system by changing unstable eigenvalues (those with non-negative real parts) into stable eigenvalues (those with negative real parts).

Extensive research has been conducted in the area of state feedback for certain classes of linear continuous systems [3]-[5], [17], [27]. Nguyen developed a state feedback technique to transform linear time-varying continuous systems which satisfy the property of lexicografixed controllability into equivalent time-invariant systems for which the eigenvalues can be arbitrarily assigned [21]. State feedback techniques have also been developed for linear time-invariant discrete systems which satisfy the property of complete state controllability [25]. However, in the case of linear time-varying discrete systems, state feedback techniques have not been developed. Now that the concept of controllability has been extended to this class of systems, state feedback techniques can also be developed. This chapter focuses on the development of state feedback techniques for a class of linear time-varying discrete systems.

5.2. Problem Statement

In order to stabilize a linear time-varying discrete system, state feedback techniques must be developed. The following statement defines the specific problem to be solved in this chapter.

Given a linear time-varying discrete system which satisfies the property of complete state controllability, develop a state feedback technique for which the closed-loop feedback system is equivalent to an asymptotically stable time-invariant system for which the eigenvalues can be arbitrarily assigned.

The scope of the problem addressed is reduced by considering only the class of linear time-varying single-input-single-output discrete systems which have two state variables. The approach taken to solve this problem is similar to that taken in [21] for linear time-varying continuous systems. First, in section 5.3, a canonical transformation is derived which transforms the state equation of the given system into an equivalent canonical state equation. Then, in section 5.4, a state feedback method is developed which transforms the canonical state equation into an equivalent time-invariant state equation for which the eigenvalues can be arbitrarily assigned. Eigenvalues which asymptotically stabilize the system are then assigned. These results are illustrated through the use of an example in section 5.5.

5.3. Canonical Transformation for a Class of Linear Time-Varying Discrete Systems

Three steps are necessary in the development of a canonical transformation for the class of linear time-varying single-input-single-output discrete systems which have two state variables:

1) The concept of an equivalence transformation for this class of systems must be defined.

2) Equations describing the equivalent state and input matrices must be derived.

3) An equivalence transformation converting the given system into the desired canonical form must be derived.

5.3.1. Definition of Equivalence Transformation

Consider a linear time-varying single-input-single-output discrete system which has two state variables. Such a system can be modeled using the following state equation,

$$x(k+1) = F(k)x(k) + G(k)u(k). \tag{5-1}$$

The state and input matrices of (5-1) are represented by

$$F(k) = \begin{bmatrix} f_{00}(k) & f_{01}(k) \\ f_{10}(k) & f_{11}(k) \end{bmatrix} \tag{5-2}$$

and

$$G(k) = \begin{bmatrix} g_0(k) \\ g_1(k) \end{bmatrix}. \tag{5-3}$$

If there exists a nonsingular matrix Q(k) such that

$$\overline{x}(k) = Q(k)x(k), \tag{5-4}$$

then the system described by

$$\bar{x}(k+1) = \bar{F}(k)\bar{x}(k) + \bar{G}(k)u(k)$$

(5-5)

is said to be equivalent to the original system (5-1) and the matrix Q(k) is said to be an equivalence transformation.

5.3.2. Derivation of the Equivalent State and Input Matrices

The derivation begins by calculating the equivalent state vector (5-4) at the k+1th sampling instant:

$$\bar{x}(k+1) = Q(k+1)x(k+1).$$

(5-6)

Substituting the discrete state equation (5-1) into (5-6) yields

$$\begin{aligned}\bar{x}(k+1) &= Q(k+1)\left[F(k)x(k) + G(k)u(k)\right] \\ &= Q(k+1)F(k)x(k) + Q(k+1)G(k)u(k).\end{aligned}$$

(5-7)

Solving the equivalent state vector (5-4) for x(k) and substituting the result into (5-7) yields

$$\bar{x}(k+1) = Q(k+1)F(k)Q^{-1}(k)\bar{x}(k) + Q(k+1)G(k)u(k).$$

(5-8)

The equivalent state equation (5-5) can be obtained by substituting

$$\bar{F}(k) = Q(k+1)F(k)Q^{-1}(k)$$

(5-9)

and

$$\overline{G}(k) = Q(k+1)G(k)$$

(5-10)

into (5-8). Equations (5-9) and (5-10) define the equivalent state and input matrices.

5.3.3. Derivation of the Equivalence Transformation

In order to derive an equivalence transformation, it is necessary to first identify the desired canonical form. For this derivation, the discrete equivalent of the canonical form obtained by the application of Nguyen's second canonical transformation (2-32) is selected as the desired canonical form. This equivalence transformation converts the given discrete system (5-1) into an equivalent system in which the state and input matrices have the following canonical form,

$$\overline{F}(k) = \begin{bmatrix} \bar{f}_{00}(k) & \bar{f}_{01}(k) \\ 1 & 0 \end{bmatrix}$$

(5-11)

and

$$\overline{G}(k) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(5-12)

Using (4-10), the controllability matrix of (5-1) is computed as

$$S = \begin{bmatrix} G(n-1) & F(n-1)G(n-2) \end{bmatrix}.$$

(5-13)

Substituting k=n into (5-13) yields,

$$\overline{S}(k)=\begin{bmatrix} g_0(k-1) & f_{00}(k-1)g_0(k-2)+f_{01}(k-1)g_1(k-2) \\ g_1(k-1) & f_{10}(k-1)g_0(k-2)+f_{11}(k-1)g_1(k-2) \end{bmatrix}$$
$$=\begin{bmatrix} \overline{S}_{00}(k) & \overline{S}_{01}(k) \\ \overline{S}_{10}(k) & \overline{S}_{11}(k) \end{bmatrix}.$$

(5-14)

The inverse of (5-14) is calculated as

$$\overline{S}^{-1}(k)=\frac{1}{\overline{S}_{00}(k)\overline{S}_{11}(k)-\overline{S}_{10}(k)\overline{S}_{01}(k)}\begin{bmatrix} \overline{S}_{11}(k) & -\overline{S}_{01}(k) \\ -\overline{S}_{10}(k) & \overline{S}_{00}(k) \end{bmatrix}.$$

(5-15)

Defining the row vector $\beta(k)$ as the bottom row of (5-15) yields,

$$\beta(k)=\begin{bmatrix} \frac{-\overline{S}_{10}(k)}{\overline{S}_{00}(k)\overline{S}_{11}(k)-\overline{S}_{10}(k)\overline{S}_{01}(k)} & \frac{\overline{S}_{00}(k)}{\overline{S}_{00}(k)\overline{S}_{11}(k)-\overline{S}_{10}(k)\overline{S}_{01}(k)} \end{bmatrix}=\begin{bmatrix} \beta_0(k) & \beta_1(k) \end{bmatrix}.$$

(5-16)

The desired equivalence transformation Q(k) is then defined as

$$Q(k)=\begin{bmatrix} Q_{00}(k) & Q_{01}(k) \\ Q_{10}(k) & Q_{11}(k) \end{bmatrix}.$$

(5-17)

The inverse of (5-17) is computed as

$$Q^{-1}(k)=\begin{bmatrix} \frac{Q_{11}(k)}{Q_{00}(k)Q_{11}(k)-Q_{01}(k)Q_{10}(k)} & \frac{-Q_{01}(k)}{Q_{00}(k)Q_{11}(k)-Q_{01}(k)Q_{10}(k)} \\ \frac{-Q_{10}(k)}{Q_{00}(k)Q_{11}(k)-Q_{01}(k)Q_{10}(k)} & \frac{Q_{00}(k)}{Q_{00}(k)Q_{11}(k)-Q_{01}(k)Q_{10}(k)} \end{bmatrix}.$$

(5-18)

Substituting (5-9) into (5-11) and (5-10) into (5-12) yields

$$\overline{F}(k) = Q(k+1)F(k)Q^{-1}(k) = \begin{bmatrix} \bar{f}_{00}(k) & \bar{f}_{01}(k) \\ 1 & 0 \end{bmatrix}$$

(5-19)

and

$$\overline{G}(k) = Q(k+1)G(k) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(5-20)

The two elements in the bottom row of (5-19) and the two elements of (5-20) represent the following set of four simultaneous linear equations,

$$\frac{\begin{matrix} Q_{11}(k)Q_{10}(k+1)f_{00}(k) + Q_{11}(k)Q_{11}(k+1)f_{10}(k) \\ -Q_{10}(k)Q_{10}(k+1)f_{01}(k) - Q_{10}(k)Q_{11}(k+1)f_{11}(k) \end{matrix}}{Q_{00}(k)Q_{11}(k) - Q_{01}(k)Q_{10}(k)} = 1,$$

(5-21)

$$\frac{-\begin{pmatrix} Q_{01}(k)Q_{10}(k+1)f_{00}(k) + Q_{01}(k)Q_{11}(k+1)f_{10}(k) \\ -Q_{00}(k)Q_{10}(k+1)f_{01}(k) - Q_{00}(k)Q_{11}(k+1)f_{11}(k) \end{pmatrix}}{Q_{00}(k)Q_{11}(k) - Q_{01}(k)Q_{10}(k)} = 0,$$

(5-22)

$$Q_{00}(k)g_0(k-1) + Q_{01}(k)g_1(k-1) = 1,$$

(5-23)

and

$$Q_{10}(k)g_0(k-1) + Q_{11}(k)g_1(k-1) = 0.$$

(5-24)

Solving (5-21) and (5-22) for $Q_{00}(k)$ and $Q_{01}(k)$ yields

$$Q_{00}(k) = f_{00}(k)Q_{10}(k+1) + f_{10}(k)Q_{11}(k+1)$$

(5-25)

and

$$Q_{01}(k) = f_{01}(k)Q_{10}(k+1) + f_{11}(k)Q_{11}(k+1).$$

(5-26)

Substituting (5-25) and (5-26) into (5-23) and then solving (5-23) and (5-24) for $Q_{10}(k)$ and $Q_{11}(k)$ yields

$$\begin{aligned} Q_{10}(k) &= \frac{-g_1(k-1)}{g_0(k-1)\begin{bmatrix} f_{10}(k-1)g_0(k-2) \\ +f_{11}(k-1)g_1(k-2) \end{bmatrix} - g_1(k-1)\begin{bmatrix} f_{00}(k-1)g_0(k-2) \\ +f_{01}(k-1)g_1(k-2) \end{bmatrix}} \\ &= \frac{-\overline{S}_{10}(k)}{\overline{S}_{00}(k)\overline{S}_{11}(k) - \overline{S}_{10}(k)\overline{S}_{01}(k)} \\ &= \beta_0(k) \end{aligned}$$

(5-27)

and

$$\begin{aligned} Q_{11}(k) &= \frac{g_0(k-1)}{g_0(k-1)\begin{bmatrix} f_{10}(k-1)g_0(k-2) \\ +f_{11}(k-1)g_1(k-2) \end{bmatrix} - g_1(k-1)\begin{bmatrix} f_{00}(k-1)g_0(k-2) \\ +f_{01}(k-1)g_1(k-2) \end{bmatrix}} \\ &= \frac{\overline{S}_{00}(k)}{\overline{S}_{00}(k)\overline{S}_{11}(k) - \overline{S}_{10}(k)\overline{S}_{01}(k)} \\ &= \beta_1(k). \end{aligned}$$

(5-28)

By combining the results of (5-25)-(5-28), the equivalence transformation Q(k) is defined as

$$Q(k) = \begin{bmatrix} \beta(k+1)F(k) \\ \beta(k) \end{bmatrix}.$$

(5-29)

This derivation results in a new equivalence transformation (5-29) which is dependent on the discrete controllability matrix (4-10). Since the equivalent state matrix (5-19) is a function of the inverse of the controllability matrix, complete state controllability is a necessary condition for the equivalence transformation Q(k) to exist.

5.4. State Feedback Techniques for a Class of Linear Time-Varying Discrete Systems

Now that an equivalence transformation has been derived for linear time-varying discrete systems, it is possible to develop a state feedback theorem for this class of systems. A new state feedback theorem for this class of systems is now introduced and subsequently proven.

Theorem 5.1

Consider a linear time-varying single-input-single-output discrete system which has two state variables. If the state equation (5-1) of the given system is completely state controllable, then there exists a state feedback law u(k)=K(k)x(k) such that the closed-loop feedback system is equivalent to an asymptotically stable time-invariant system whose state matrix assumes the form of (5-30) with any desired eigenvalues.

$$\overline{F}_c = \begin{bmatrix} -\alpha_1 & -\alpha_0 \\ 1 & 0 \end{bmatrix}. \tag{5-30}$$

A simple proof of Theorem 5.1 is presented. In the previous section it was shown that, if the state equation (5-1) of a linear time-varying single-input-single-output discrete system which has two state variables is completely state controllable, then there exists a canonical transformation (5-29) which converts (5-1) into an equivalent state equation with

state and input matrices given by (5-11) and (5-12). If the desired eigenvalues of the closed loop feedback equivalent system are represented as λ_1 and λ_2, then the following expression can be formed:

$$(\lambda-\lambda_1)(\lambda-\lambda_2)=\lambda^2+\alpha_1\lambda+\alpha_0. \tag{5-31}$$

If the equivalent feedback matrix is chosen as

$$\overline{K}(k)=\left[-\bar{f}_{00}(k)-\alpha_1 \quad -\bar{f}_{01}(k)-\alpha_0\right], \tag{5-32}$$

then the closed loop feedback equivalent system will have the following form,

$$\begin{aligned}
\overline{F}_c &= \overline{F}(k)+\overline{G}(k)\overline{K}(k)\\
&=\begin{bmatrix}\bar{f}_{00}(k) & \bar{f}_{01}(k)\\ 1 & 0\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}\left[-\bar{f}_{00}(k)-\alpha_1 \quad -\bar{f}_{01}(k)-\alpha_0\right]\\
&=\begin{bmatrix}\bar{f}_{00}(k) & \bar{f}_{01}(k)\\ 1 & 0\end{bmatrix}+\begin{bmatrix}-\bar{f}_{00}(k)-\alpha_1 & -\bar{f}_{01}(k)-\alpha_0\\ 0 & 0\end{bmatrix}\\
&=\begin{bmatrix}-\alpha_1 & -\alpha_0\\ 1 & 0\end{bmatrix}.
\end{aligned} \tag{5-33}$$

The eigenvalues of (5-33) are the desired eigenvalues and the solution to (5-31). The proof is complete.

To obtain the state feedback gain matrix, it is necessary to multiply the equivalent feedback matrix (5-32) by the equivalence transformation (5-29). That is,

$$K(k)=\overline{K}(k)Q(k). \tag{5-34}$$

If the matrix K(k) is chosen correctly, the closed-loop system represented in Figure 5-1 is asymptotically stable.

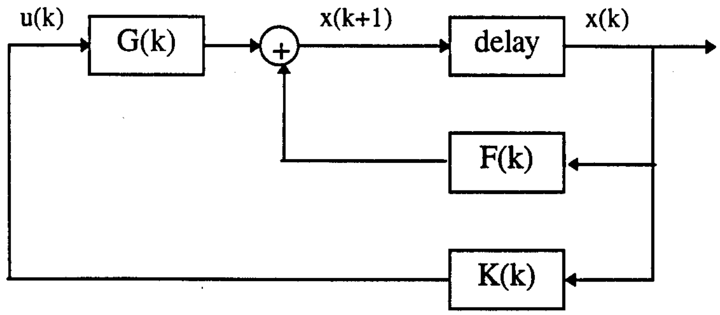

Figure 5-1 Asymptotically Stabilized, Closed-Loop System

An algorithm for determining the required feedback matrix K(k) is given below.

Algorithm 5.1

1) Compute the equivalence transformation Q(k).

2) Compute the equivalent state and input matrices.

3) Select desired poles and compute the feedback matrix K(k) required to give the desired poles.

5.5. Example

The following example is used to illustrate the validity of Theorem 5.1 and to demonstrate the use of Algorithm 5.1 to asymptotically stabilize a linear time-varying discrete system.

Consider a linear time-varying single-input-single-output discrete system with state equation given by (3-22). Since it was shown in example 4-1 that this system is completely state controllable, then by Theorem 5.1 there exists a state feedback law to asymptotically

stabilize the system. Algorithm 5.1 is now employed to determine the required feedback gain matrix K(k).

a) **Compute the equivalence transformation Q(k).** Using (5-14), the following matrix is formed,

$$\overline{S}(k) = \begin{bmatrix} \begin{pmatrix} 0.4881231108\exp(-1.5k) \\ +0.39334693403 \end{pmatrix} & \begin{pmatrix} 2.494549750\exp(-1.5k) \\ +0.2386512185 \end{pmatrix} \\ 1.359140915\exp(-0.5k) & 1.359140915\exp(-0.5k) \end{bmatrix}. \tag{5-35}$$

The inverse of (5-35) is computed as

$$\overline{S}^{-1}(k) = \frac{\begin{bmatrix} -1.359140914 & 2.494549750\exp(-k) + 0.2386512185\exp(0.5k) \\ 1.359140914 & -0.4881231108\exp(-k) - 0.3934693403\exp(0.5k) \end{bmatrix}}{2.727016538\exp(-1.5k) - 0.2104196433}. \tag{5-36}$$

The matrix β(k) is defined in (5-16) as the bottom row of the inverse of $\overline{S}(k)$. Computing β(k) yields

$$\beta(k) = \frac{\begin{bmatrix} 1.359140914 & -0.4881231108\exp(-k) - 0.3934693403\exp(0.5k) \end{bmatrix}}{2.727016538\exp(-1.5k) - 0.2104196433}, \tag{5-37}$$

The equivalence transformation Q(k) is computed from (5-29) as

$$Q(k)=\begin{bmatrix} \dfrac{1}{\begin{pmatrix}0.7381231109\exp(-1.5k)\\ -0.25525193\end{pmatrix}} & \dfrac{0.151632665\exp(-k)-0.3934693403}{0.6084796369\exp(-1.5k)-0.2104196433} \\ \dfrac{1.359140914}{\begin{pmatrix}2.727016538\exp(-1.5k)\\ -0.2104196433\end{pmatrix}} & \dfrac{\begin{pmatrix}-0.4881231108\exp(-k)\\ -0.3934693403\exp(0.5k)\end{pmatrix}}{2.727016538\exp(-1.5k)-0.2104196433} \end{bmatrix}.$$

(5-38)

b) **Compute the equivalent state and input matrices.** From (5-9), the equivalent state matrix is computed as

$$\overline{F}(k)=\begin{bmatrix}\bar{f}_{00}(k) & \bar{f}_{01}(k)\\ 1 & 0\end{bmatrix},$$

(5-39)

where

$$\bar{f}_{00}(k)=\frac{1.023932728\exp(-2.5k)-1.039734596\exp(-k)+0.2371047881\exp(0.5k)}{0.2753779232\exp(-2.5k)-0.52201592213\exp(-k)+0.1475880879\exp(0.5k)}$$

(5-40)

and

$$\bar{f}_{01}(k)=\frac{-0.7485548044\exp(-2.5k)+1.217886384\exp(-k)-0.08951670022\exp(0.5k)}{0.2753779232\exp(-2.5k)-0.52201592213\exp(-k)+0.1475880879\exp(0.5k)}.$$

(5-41)

From (5-10), the equivalent input matrix is computed as

$$\overline{G}(k)=\begin{bmatrix}1\\ 0\end{bmatrix}.$$

(5-42)

c) **Calculate the feedback matrix K(k) required to give desired poles at**

$$\lambda_1, \lambda_2 = 0.5 \pm 0.5i.$$

(5-43)

The characteristic equation of the desired system is computed from (5-31) as

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 + \alpha_1 \lambda + \alpha_0,$$

(5-44)

where

$$\alpha_0 = 0.5$$

(5-45)

and

$$\alpha_1 = -1.$$

(5-46)

The equivalent feedback matrix is computed from (5-32) as

$$\overline{K}(k) = \left[-\bar{f}_{00}(k) + 1 \quad -\bar{f}_{01}(k) - 0.5\right] = \left[\bar{k}_0(k) \quad \bar{k}_1(k)\right],$$

(5-47)

where

$$\bar{k}_0(k) = \frac{-7.485548048\exp(-2.5k) + 5.177186739\exp(-k) - 0.8951670020\exp(0.5k)}{2.753779232\exp(-2.5k) - 5.2201592213\exp(-k) + 1.475880879\exp(0.5k)}$$

(5-48)

and

$$\bar{k}_1(k) = \frac{6.10865843\exp(-2.5k) - 9.56878423\exp(-k) + 0.1572265627\exp(0.5k)}{2.753779232\exp(-2.5k) - 5.2201592213\exp(-k) + 1.475880879\exp(0.5k)}.$$

(5-49)

The closed loop feedback equivalent system is computed from (5-33) as

$$\bar{F}_c = \bar{F}(k) + \bar{G}(k)\bar{K}(k) = \begin{bmatrix} 1 & -0.5 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -\alpha_1 & -\alpha_0 \\ 1 & 0 \end{bmatrix}.$$

(5-50)

Equation (5-50) shows that the time-invariant equivalent matrix has the desired eigenvalues (5-43) and is therefore asymptotically stable. The feedback matrix K(k) is calculated from (5-34) as

$$K(k) = \begin{bmatrix} k_0(k) & k_1(k) \end{bmatrix},$$

(5-51)

where

$$k_0(k) = \frac{\begin{pmatrix} 117.7593053\exp(-4k) - 32.76513858\exp(-2.5k) \\ +0.43815282\exp(-k) - 1.103119477\exp(0.5k) \end{pmatrix}}{\begin{pmatrix} -45.69439598\exp(-5.5k) + 105.9473859\exp(-4k) \\ -62.34696063\exp(-2.5k) + 12.669827685\exp(-k) - 0.6534673094\exp(0.5k) \end{pmatrix}}$$

(5-52)

and

$$k_1(k) = \frac{\begin{pmatrix} -49.09660744\exp(-5k) + 124.1855694\exp(-3.5k) \\ -49.43022168\exp(-2k) + 6.039829596\exp(-0.5k) - 0.6109680328\exp(k) \end{pmatrix}}{\begin{matrix} 45.69439598\exp(-5.5k) - 105.94738589\exp(-4k) + 62.34696063\exp(-2.5k) \\ -12.669827685\exp(-k) + 0.6534673094\exp(0.5k) \end{matrix}}.$$

(5-53)

5.6. Conclusion

Until now, equivalence transformation, state feedback, and stabilization techniques had not been developed for linear time-varying single-input-single-output discrete systems. The results of the original research in these areas for this class of systems in which there are two state variables are presented in this chapter. A new equivalence transformation which converts such a system into an equivalent canonical system is derived in section 5.3. A new state feedback theorem for this class of systems is developed and proven in section 5.4. A new stabilization algorithm which determines the required feedback gain matrix to asymptotically stabilize such a system is also presented in section 5.4. The use of the equivalence transformation, state feedback theorem, and stabilization algorithm is demonstrated in section 5.5 through the use of an example. The example successfully demonstrates that a linear time-varying discrete system which is completely state controllable can be asymptotically stabilized through the use of state feedback.

CHAPTER 6
CONCLUSION

6.1. Summary

This report presents the development of a new control scheme for linear time-varying continuous systems which do not satisfy the property of lexicografixed controllability. The scheme involves the application of sample and hold techniques for discretizing, and canonical transformation and state feedback techniques for stabilizing the given continuous system. Through the application of these methods, a linear time-varying system that is not lexicografixedly controllable in continuous time can be controlled and stabilized in discrete time. Until now, the concepts of controllability and stability and the techniques of discretization, canonical transformation and state feedback had not been exploited for the class of linear time-varying discrete systems.

Chapter 3 is devoted to the derivation of equations modeling the discretization of a linear time-varying continuous system. The equations derived model the resulting discretized state equations in terms of the associated continuous state equations. The results indicate that the discretized system can be modeled as long as the state transition matrix of the continuous system can be computed.

In Chapter 4, the concept of controllability is investigated for the class of linear time-varying multivariable discrete systems. Using the definition of controllability, a controllability matrix is derived and the controllability theorem developed. It is proved that the full rank condition of the controllability matrix is a necessary and sufficient condition for the system to be completely state controllable. The results show that the controllability

matrix of a time-varying discrete system is actually time-invariant. This implies that in discrete time the concepts of complete state controllability, uniform controllability, and lexicografixed controllability are equivalent.

The concept of stability for linear time-varying discrete systems is investigated in Chapter 5. The derivation of a canonical transformation for linear time-varying single-input-single-output discrete systems is presented. The results show that complete state controllability is a necessary condition for the canonical transformation to exist. A state feedback technique and its associated theorem for this class of systems are also developed. The results show that state feedback can be used to asymptotically stabilize this class of systems as long as the given system is completely state controllable. An algorithm for determining the feedback matrix required to obtain the desired poles is presented.

The results of this research show that a limited class of linear time-varying continuous systems which are not lexicografixedly controllable can be controlled through the use of the proposed control scheme. Based on the results of Chapters 3-5, this class of systems can be identified as linear time-varying single-input-single-output continuous systems which have two state variables and a known state transition matrix and which satisfy the property of complete state controllability in discrete time.

Although the problem statement delineated in Chapter 1 focuses on linear time-varying multivariable continuous systems which are not lexicografixedly controllable, the results of this research are valid for other classes of systems as well. The discretization equations in Chapter 3 are valid for all continuous systems with a known state transition matrix. The controllability matrix and theorem presented in Chapter 4 are valid for all

discrete systems. The canonical transformation and state feedback techniques presented in Chapter 5 are valid for any linear time-varying single-input-single-output discrete system which has two state variables.

6.2. Recommendations for Future Research

This research effort solves the problems of controllability and stabilization for a limited class of linear discrete systems. Extensions to this work can be made in several directions. Some possibilities for future research are listed below:

1) Using the discretization results of Chapter 3 and the controllability results of Chapter 4, determine the subclass of linear time-varying continuous systems which are not lexicografixedly controllable that can be made completely state controllable through discretization.
2) Investigate the use of a variable sampling period on the complete state controllability of discretized systems.
3) Using the duality characteristic between controllability and observability, derive an observability matrix and develop the associated observability theorem for linear time-varying multivariable discrete systems.
4) Extend the use of the canonical transformation derived in Chapter 5 to cover the entire class of linear time-varying multivariable discrete systems which are completely state controllable. Develop other canonical transformations for this class of systems. Using the duality characteristic between controllability and observability, derive canonical transformations for linear time-varying multivariable discrete systems which are completely state observable.

5) Extend the state feedback technique developed in Chapter 5 to cover the entire class of linear time-varying multivariable discrete systems which are completely state controllable. Using the duality characteristic between controllability and observability, develop an asymptotic state estimator technique for linear time-varying multivariable discrete systems which are completely state observable.

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APPENDIX A
ILIR SOFTWARE ALGORITHMS

This Appendix contains the symbolic math routines written in support of this research effort. The control algorithms developed in Chapters 3-5 were coded in the Maple symbolic math language and stored in a library package which is included in Section A.1. The Maple code written to solve the example problems in Chapters 3-5 is included in Section A.2. The example problems are solved by invoking the procedures stored in the ILIR library package.

A.1. Maple Library Routines

```
#    ILIR package.
#    Programmed by Mark G. Matthews
#    CECOM RD&E Center,
#    Power Generation Branch
#    AMSEL-RD-C2-PP-P, Fort Belvoir, VA. 22060
#    This package consists of the Maple code written in support of the FY94 In-House
#    Laboratory Independent Research (ILIR) Topic,
#    "State Controllability Techniques For Linear Time-Varying Discrete Systems."
```

```
#    The following routine is used in Chapter 3 to compute the controllability matrix of a
#    linear time-varying continuous system.
#
ilir[ContControlMatrix]:=proc(`A(t)`,`B(t)`)
local `M1(t)`, `M2(t)`,`M3(t)`: # local variable declarations
global `M(t)`: # global variable declarations
if
coldim(`A(t)`)>3 # the algorithm was only written for 2x2 and 3x3 state matrices
then
print(`this algorithm will not work for systems with a state matrix bigger than 3x3`);
RETURN
fi:
`M1(t)`:=evalm(`B(t)`):
`M2(t)`:=evalm(-`A(t)`&*`M1(t)`+map(diff,`M1(t)`,t)):
if coldim(`A(t)`)=2
then `M(t)`:=augment(`M1(t)`,`M2(t)`); # eq (2-18) for a 2x2 state matrix
RETURN
else
`M3(t)`:=evalm(-`A(t)`&*`M2(t)`+map(diff,`M2(t)`,t)):
`M(t)`:=augment(`M1(t)`,`M2(t)`, `M3(t)`); # eq (2-18) for a 3x3 state matrix
fi:
end:
```

```
#   The following routine is used in Chapter 3 to discretize a linear time-varying continuous
#   system.
#
ilir[discretize]:=proc(`A(t)`,`B(t)`,`C(t)`,`D(t)`)
global `F(k,T)`, `G(k,T)`, `C(k,T)`, `D(k,T)`, `int_A(tau)`, `Phi(t,t0)`, `A(tau)`:
`A(tau)`:=subs(t=tau,evalm(`A(t)`));
`int_A(tau)`:=map(int,`A(tau)`,tau=t0..t);
if      # check for commutative property
equal(evalm(`A(t)`&*`int_A(tau)`),evalm(`int_A(tau)`&*`A(t)`))=false
then  # algorithm written to handle commutative systems only
print(`this algorithm will not work for systems with a non-commutative state matrix`);
RETURN;
else
`Phi(t,t0)`:=exponential(`int_A(tau)`):   # eq (2-9)
`F(k,T)`:=map(simplify,subs(t=(k+1)*T,t0=k*T,evalm(`Phi(t,t0)`))): # eq (3-9)
`G(k,T)`:=map(int,evalm(subs(t=(k+1)*T,t0=tau,evalm(`Phi(t,t0)`))
        &*subs(t=tau,evalm(`B(t)`))),tau=(k*T)..(k+1)*T):  # eq (3-10)
`G(k,T)`:=map(simplify,evalm(`G(k,T)`)):
`C(k,T)`:=subs(t=k*T,evalm(`C(t)`)):  # eq (3-11)
`D(k,T)`:=subs(t=k*T,evalm(`D(t)`)):  # eq (3-12)
fi;
end:
```

```
#    The following routine is used in Chapter 4 to compute the controllability matrix of a
#    linear time-varying discrete system.
#
ilir[DiscreteControlMatrix]:=proc(`F(k,T)`,`G(k,T)`)
global `S(T)`,Sdet:
local G2, G1, G0, F2, F1, S2, S1,S0:
if
coldim(`F(k,T)`)>3  # the algorithm was only written for 2x2 and 3x3 state matrices
then
print(`this algorithm will not work for systems with a state matrix bigger than 3x3`);
RETURN
fi:
if
coldim(`F(k,T)`)=2
then
G1:=subs(k=1,evalm(`G(k,T)`)):
G0:=subs(k=0,evalm(`G(k,T)`)):
F1:=subs(k=1,evalm(`F(k,T)`)):
S1:=evalm(F1&*G0):
S0:=evalm(G1):
`S(T)`:=map(simplify,augment(S0,S1)):  # eq (4-10) for a 2x2 state matrix
```

```
if
coldim(`S(T)`)=2
then
Sdet:=simplify(det(`S(T)`)); # determinant of the discrete controllability matrix
fi:
RETURN
else
G2:=map(simplify,subs(k=2,evalm(`G(k,T)`)));
G1:=map(simplify,subs(k=1,evalm(`G(k,T)`)));
G0:=map(simplify,subs(k=0,evalm(`G(k,T)`)));
F2:=map(simplify,subs(k=2,evalm(`F(k,T)`)));
F1:=map(simplify,subs(k=1,evalm(`F(k,T)`)));
S0:=evalm(G2);
S1:=map(simplify,evalm(F2&*G1));
S2:=map(simplify,evalm(F2&*F1&*G0));
`S(T)`:=augment(S0,S1,S2); # eq (4-10) for a 3x3 state matrix
fi:
end:
```

```
#   The following routine is used in Chapter 4 to compute the inputs required to transfer the
#   controllable system from a given state to a desired state in n sampling periods.
#
ilir[InputsRequired]:=proc(x0,xfinal,period)
global `u(required)`, `x(k+1)`, `x(1)`, `x(2)`, `x(3)`, `F(k)`, `G(k)`, eq1, eq2, eq3:
if
rowdim(xfinal)>3  # the algorithm was only written for 2x2 and 3x3 state matrices
then
print(`this algorithm will not work for systems with a state matrix bigger than 3x3`);
RETURN
fi:
`F(k)`:=map(evalf,map(expand,subs(T=period,evalm(`F(k,T)`))));
# the state matrix at a fixed sampling period
`G(k)`:=map(evalf,map(expand,subs(T=period,evalm(`G(k,T)`))));
#  the input matrix at a fixed sampling period
if
rowdim(xfinal)=2  #  algorithm for a system with a 2x2 state matrix
then
`x(k+1)`:=evalm(`F(k)`)&*`x(k)`+evalm(`G(k)`)*`u(k)`:
# the state equation at a fixed sampling period
`x(1)`:=evalm(subs(k=0,`x(k)`=evalm(x0),`u(k)`=u0,`x(k+1)`)):
#  the state equation at the first sampling instant
```

```
`x(2)`:=evalm(subs(k=1,`x(k)`=`x(1)`,`u(k)`=u1,`x(k+1)`)):

# the state equation at the second sampling instant

eq1:=`x(2)`[1,1]: # first simultaneous linear equation

eq2:=`x(2)`[2,1]: # second simultaneous linear equation

`u(required)`:=evalf(solve({eq1=xfinal[1,1],eq2=xfinal[2,1]},{u0,u1}));

# solving two simultaneous linear equation with two unknowns to determine the input

# required to transfer the system from the initial state to the desired state

RETURN

fi:

if

rowdim(xfinal)=3 # algorithm for a system with a 3x3 state matrix

then

`x(k+1)`:=evalm(`F(k)`)&*`x(k)`+evalm(`G(k)`)&*`u(k)`;

# the state equation at a fixed sampling period

`x(1)`:=map(evalf,evalm(subs(k=0,`x(k)`=evalm(x0),`u(k)`=matrix(2,1,[u00,u10]),`x(k+1)`))

);

# the state equation at the first sampling instant

`x(2)`:=map(evalf,evalm(subs(k=1,`x(k)`=evalm(`x(1)`),`u(k)`=matrix(2,1,[u01,u11]),

        `x(k+1)`)));

# the state equation at the second sampling instant

`x(3)`:=map(evalf,evalm(subs(k=2,`x(k)`=evalm(`x(2)`),`u(k)`=matrix(2,1,[u02,u12]),

        `x(k+1)`)));
```

```
# the state equation at the third sampling instant

eq1:=`x(3)`[1,1]=xfinal[1,1]; # first simultaneous linear equation

eq2:=`x(3)`[2,1]=xfinal[2,1]; # second simultaneous linear equation

eq3:=`x(3)`[3,1]=xfinal[3,1]; # third simultaneous linear equation

`u(required)`:=solve({eq1,eq2,eq3},{u00,u01,u02,u10,u11,u12}):

# solving three simultaneous linear equations with six unknowns to find the non-unique input

# required to transfer the system from the initial state to the final desired state

RETURN

fi:

end:
```

```
#    The following example is used in Chapter 5 to compute the feedback gain matrix
required
#    to give closed-loop poles at desired locations.
#
ilir[StateFeedback]:=proc(pole1,pole2,period)
global `Q(k)`, `Fbar(k)`, `Gbar(k)`, `Fbar(k)c`,`K(k)`,`F(k)`,`G(k)`,`Kbar(k)`,`Sbar(k)`,
        `Sbarinv(k)`, `Beta(k)`, char_eq, alpha, alpha0, alpha1, alpha2:
local `Sbar00(k)`, `Sbar01(k)`, `Sbar10(k)`, `Sbar11(k)`, `Beta(k+1)F(k)`:
`F(k)`:=map(evalf,map(expand,subs(T=period,evalm(`F(k,T)`))));
`G(k)`:=map(evalf,map(expand,subs(T=period,evalm(`G(k,T)`))));
`Sbar00(k)`:=evalf(expand(subs(k=k-1,`G(k)`[1,1]))):
`Sbar10(k)`:=evalf(expand(subs(k=k-1,`G(k)`[2,1]))):
`Sbar01(k)`:=evalf(expand(subs(k=k-1,`F(k)`[1,1])*subs(k=k-2,`G(k)`[1,1])
        +subs(k=k-1,`F(k)`[1,2])*subs(k=k-2,`G(k)`[2,1]))):
`Sbar11(k)`:=evalf(expand(subs(k=k-1,`F(k)`[2,1])*subs(k=k-2,`G(k)`[1,1])
        +subs(k=k-1,`F(k)`[2,2])*subs(k=k-2,`G(k)`[2,1]))):
`Sbar(k)`:=matrix(2,2,[`Sbar00(k)`,`Sbar01(k)`,`Sbar10(k)`,`Sbar11(k)`]);
`Sbarinv(k)`:=inverse(`Sbar(k)`);
`Beta(k)`:=map(evalf,map(expand,matrix(1,2,[`Sbarinv(k)`[2,1], `Sbarinv(k)`[2,2]]))):
`Beta(k+1)F(k)`:=
        map(evalf,map(expand,evalm(subs(k=k+1,evalm(`Beta(k)`))&*evalm(`F(k)`)))):
`Q(k)`:=map(simplify,matrix(2,2,[`Beta(k+1)F(k)`[1,1],`Beta(k+1)F(k)`[1,2],`Beta(k)`[1,1],
```

```
        `Beta(k)`[1,2]])):
`Fbar(k)`:=map(simplify,map(expand,evalm(subs(k=k+1,evalm(`Q(k)`))
        &*evalm(`F(k)`)&*inverse(evalm(`Q(k)`)))));
`Fbar(k)`[2,1]:=1:
`Fbar(k)`[2,2]:=0;
`Gbar(k)`:=matrix(2,1,[1,0]);
char_eq:=expand((lambda-pole1)*(lambda-pole2));
alpha0:=coeff(char_eq,lambda,0):
alpha1:= coeff(char_eq,lambda,1):
alpha2:= coeff(char_eq,lambda,2):
alpha:=coeffs(normal(expand((lambda-pole1)*(lambda-pole2))));
`Kbar(k)`:=map(simplify,matrix(1,2,[-`Fbar(k)`[1,1]-alpha1,-`Fbar(k)`[1,2]-alpha0])):
`Fbar(k)c`:=matrix(2,2,[-alpha1,-alpha0,1,0]):
`K(k)`:=map(simplify,evalm(`Kbar(k)`&*`Q(k)`)):
end:
#
save ilir;  # save the ilir library package to a file
```

A.2. Maple Code for Example Problems

A.2.1. Example 3-1

```
> # File=ilir3-1.ms

> #    This file contains the code used to solve example 3-1

> with(linalg):  # make the linear algebra package available

> `A(t)`:=matrix(2,2,[-1,exp(-2*t),0,-1]):  #  the given state matrix

> `B(t)`:=matrix(2,1,[1,exp(1-t)]):  #  the given input matrix

> `C(t)`:=matrix(1,2,[exp(-2*t),-1]):  #  the given output matrix

> `D(t)`:=1:  #  the given direct transmission matrix

> read `ilir`:  #  read the ilir library package

> ilir[ContControlMatrix](`A(t)`,`B(t)`):  #  compute the continuous controllability matrix

> evalm(`M(t)`);  #  eq (3-18)

> `M(t)`:=map(combine,`M(t)`,exp);  # simplifying eq (3-18)

> `Mdet(t)`:=combine(expand(det(`M(t)`)),exp);  # eq (3-19)

> plot(`Mdet(t)`,t=0..5);  #  Figure 3-1

> fsolve(`Mdet(t)`=0,t=0..0.5);  #  solving for the zero crossing in Figure 3-1

> ilir[discretize](`A(t)`,`B(t)`,`C(t)`,`D(t)`):  #  discretize the given system

> evalm(`int_A(tau)`&*`A(t)`);  # eq (3-20)

> print(`Phi(t,t0)`);  #  eq (3-21)

> print(`F(k,T)`);  #  eq (3-22)

> print(`G(k,T)`);  #  eq (3-23)

> print(`C(k,T)`);  #  eq (3-24)

> print(`D(k,T)`);  #  eq (3-25)
```

A.2.2. Example 3-2

```
> #   File=ilir3-2.ms
> #   This file contains the code used to solve example 3-1
> with(linalg): # make the linear algebra package available
> `A(t)`:=matrix(2,2,[exp(-t),0,0,exp(-t)]): # the given state matrix
> `B(t)`:=matrix(2,2,[1,1,1,1]): # the given input matrix
> `C(t)`:=matrix(1,2,[t,1]): # the given output matrix
> `D(t)`:=matrix(1,2,[t,1]): # the given direct transmission matrix
> read `ilir`: # read the ilir library package
> ilir[ContControlMatrix](`A(t)`,`B(t)`): # compute the continuous controllability matrix
> print(`M(t)`); # eq (3-31)
> ilir[discretize](`A(t)`,`B(t)`,`C(t)`,`D(t)`): # discretize the given system
> print(`Phi(t,t0)`); # eq (3-32)
> print(`F(k,T)`); # eq (3-33)
> print(`G(k,T)`); # eq (3-34)
> print(`C(k,T)`); # eq (3-37)
> print(`D(k,T)`); # eq (3-38)
```

A.2.3. Example 3-3

```
> # File=ilir3-3.ms

> #    This file contains the code used to solve example 3-3

> with(linalg): # make the linear algebra package available

> `A(t)`:=matrix(3,3,[2-4*exp(-2*t),0,0,0,1,0,0,0,3/2-2/(t+1)]); # the given state matrix

> `B(t)`:=matrix(3,2,[1,0,1,1,0,1]); # the given input matrix

> read `ilir`: # read the ilir library package

> ilir[ContControlMatrix](`A(t)`,`B(t)`): # compute the continuous controllability matrix

> print(`M(t)`); # eq (3-44)

> ilir[discretize](`A(t)`,`B(t)`,`C(t)`,`D(t)`): # discretize the given system

>  print(`Phi(t,t0)`); # eq (3-45)

> print(`F(k,T)`); # eq (3-46)

> print(`G(k,T)`);  # eq (3-47)
```

A.2.4. Example 4-1

```
> # File=ilir4-1.ms

> # This file contains the code used to solve example 4-1

> with(linalg): # make the linear algebra package available

> `A(t)`:=matrix(2,2,[-1,exp(-2*t),0,-1]): # the given state matrix

> `B(t)`:=matrix(2,1,[1,exp(1-t)]): # the given input matrix

> `C(t)`:=matrix(1,2,[exp(-2*t),-1]): # the given output matrix

> `D(t)`:=1: # the given direct transmission matrix

> read `ilir`: # read the ilir library package

> ilir[discretize](`A(t)`,`B(t)`,`C(t)`,`D(t)`): # discretize the given system

> ilir[DiscreteControlMatrix](`F(k,T)`,`G(k,T)`):

# calculate the discrete controllability matrix

> map(simplify,`S(T)`); # eq (4-15)

> simplify(Sdet); # eq (4-16)

> plot(Sdet,T=0..3.5); # Figure 4-1

> zero1:=fsolve(Sdet=0,T=0..0.1); # first zero crossing

> zero2:=fsolve(Sdet=0,T=0.3..0.4); # second zero crossing

> low1:=fsolve(Sdet=-0.001,T=0..0.2); # computing desirable range of T

> low2:=fsolve(Sdet=-0.001,T=0.2..0.4); # computing desirable range of T

> low3:=fsolve(Sdet=0.001,T=0.3..0.4); # computing desirable range of T

> low3:=fsolve(Sdet=0.001,T=4..5); # computing desirable range of T

> x0:=matrix(2,1,[2.0,5.0]); # given initial state
```

```
> x2:=matrix(2,1,[0.5,2.5]); # desired final state

> period:=0.5; # selected sampling period

> ilir[InputsRequired](x0,x2,period):

# calculating inputs required to transfer system from given initial state to final desired state

>print(`x(k+1)`); # eq (4-20)

> map(evalf,`x(1)`); # eq (4-21)

> map(evalf,`x(2)`); # eq (4-22)

> print(`u(required)`); # eq (4-23)
```

A.2.5. Example 4-2

```
> # File=ilir4-2.ms

> # This file contains the code used to solve example 4-2

> with(linalg): # make the linear algebra package available

> `A(t)`:=matrix(2,2,[exp(-t),0,0,exp(-t)]): # the given state matrix

> `B(t)`:=matrix(2,2,[1,1,1,1]): # the given input matrix

> `C(t)`:=matrix(1,2,[t,1]): # the given output matrix

> `D(t)`:=matrix(1,2,[t,1]): # the given direct transmission matrix

> read `ilir`: # read the ilir library package

> ilir[discretize](`A(t)`,`B(t)`,`C(t)`,`D(t)`): # discretize the given system

> ilir[DiscreteControlMatrix](`F(k,T)`,`G(k,T)`): # calculate discrete controllability matrix

> map(simplify,`S(T)`); # eq (4-24)
```

A.2.6. Example 4-3

```
> # File=ilir4-3.ms

> # This file contains the code used to solve example 4-3

> with(linalg): # make the linear algebra package available

> `A(t)`:=matrix(3,3,[2-4*exp(-2*t),0,0,0,1,0,0,0,3/2-2/(t+1)]): # the given state matrix

> `B(t)`:=matrix(3,2,[1,0,1,1,0,1]): # the given input matrix

> read `ilir`: # read the ilir library package

> ilir[discretize](`A(t)`,`B(t)`,`C(t)`,`D(t)`): # discretize the given system

> ilir[DiscreteControlMatrix](`F(k,T)`,`G(k,T)`): # calculate discrete controllability matrix

> map(simplify,`S(T)`); # eq (4-27)

> `Ssub(T)`:=augment(col(`S(T)`,1),col(`S(T)`,2),col(`S(T)`,3)); # eq (4-38)

> `Sdet(T)`:=simplify(det(`Ssub(T)`)); # eq (4-39)

> plot(`Sdet(T)`,T=0..0.5); # Figure 4-2

> first_zero:=fsolve(`Sdet(T)`=0,T,0..0.1); # first zero crossing

> second_zero:=fsolve(`Sdet(T)`=0,T,0.3..0.4); # second zero crossing

> neg1:=fsolve(`Sdet(T)`=-0.001,T,0..0.2); # computing desirable range of T

> neg2:=fsolve(`Sdet(T)`=-0.001,T,0.2..0.4); # computing desirable range of T

> pos2:=fsolve(`Sdet(T)`=0.001,T,0.35..0.4); # computing desirable range of T

> high:=fsolve(`Sdet(T)`=10^6,T,1.0..3.0); # computing desirable range of T

> x0:=matrix(3,1,[2.0,5.0,1.0]); # given initial state

> x3:=matrix(3,1,[0.5,2.5,0.]); # desired final state

> period:=0.5: # selected sampling period
```

```
> ilir[InputsRequired](x0,x3,period):

# calculating inputs required to transfer system from given initial state to final desired state

> print(`F(k)`); # eq (4-44)

> print(`G(k)`); # eq (4-45)

> print(`x(1)`); # eq (4-46)

> print(`x(2)`); # eq (4-47)

> print(`x(3)`); # eq (4-48)

> print(`u(required)`); # eq (4-49)

> u01:=2.5; # assigned input

> u02:=0;   # assigned input

> u11:=2.5; # assigned input

> solve({eq1,eq2,eq3},{u00,u10,u12}); # eq (4-51)
```

A.2.7. Example 5-1

```
# File=ilir5-1.ms

#  Maple code used to solve example 5-1

> with(linalg):  #  make the linear algebra package available

> `A(t)`:=matrix(2,2,[-1,exp(-2*t),0,-1]);  # the given state matrix

> `B(t)`:=matrix(2,1,[1,exp(1-t)]);  # the given input matrix

> `C(t)`:=matrix(1,2,[exp(-2*t),-1]);  # the given output matrix

> `D(t)`:=1;  # the given direct transmission matrix

> read `ilir`:  #  read the ilir library routines

> ilir[discretize](`A(t)`,`B(t)`,`C(t)`,`D(t)`):  #  discretize the system

> ilir[DiscreteControlMatrix](`F(k,T)`,`G(k,T)`):  #  compute discrete controllability matrix

> period:=0.5:  #  selected sampling period

> pole1:=0.5+0.5*I:  #  desired pole

> pole2:=0.5-0.5*I:  #  desired pole

> ilir[StateFeedback](pole1,pole2,period):  #  invoke the state feedback algorithm

> map(simplify,`Sbar(k)`);  # eq (5-35)

> print(`Sbarinv(k)`);  # eq (5-36)

> print(`Beta(k)`);  # eq (5-37)

> print(`Q(k)`);  # eq (5-38)

> print(`Fbar(k)`);  # eq (5-39)

> print(`Gbar(k)`);  # eq (5-42)

> print(char_eq);  # eq (5-44)
```

```
> print(alpha0, alpha1);  # eqs (5-45)-5-46)

> print(`Kbar(k)`);  # eq (5-47)

> print(`Fbar(k)c`);  # eq (5-50)

> print(`K(k)`);  # eq (5-51)
```

APPENDIX B
ISCA PAPER

The principal investigator presented an invited paper documenting interim ILIR results on controllability and served as a Session Chair at the 1994 International Conference on Computers and Their Applications [15]. This paper is included in this Appendix.

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A New Approach To Control Of Time-Varying Robotic Systems

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Abstract

A robot manipulator can be modeled by a linear, time-varying, multivariable system which is obtained by linearizing the manipulator dynamics about a moving operating point on the path that the manipulator tracks. It is found that if a time-varying system is lexicografixedly controllable, then there exists a state feedback controller which can assign an arbitrary set of eigenvalues to the closed-loop, feedback system. This paper presents some preliminary results of a new approach to control of robotic systems modeled by a linear, time-varying system that is not lexicografixedly controllable. The proposed approach considers discretization of the time-varying system and selection of a sampling time to make the resulting discrete system become lexicografixedly controllable. After defining the controllability of time-varying, discrete systems, a theorem is given and proved. An example of a single-input-single-output (SISO) system is carried out to demonstrate the dependence of controllability on sampling times.

1 Introduction

The inherent nonlinearity of robotic systems, caused by intercoupling of joints, combined with the uncertainty in their dynamics make the control of such systems a challenging task. Consequently, advanced control schemes, such as adaptive control schemes [1], have been developed for controlling robot motion. A simple but effective method for robot control is the use of linearization about a selected fixed operating point resulting in a linear, time-invariant system representing an approximated model of the robot system [8]. The performance of the above scheme starts to degrade as the robot moves away from the operating point. To improve the performance of the above scheme, a so called *path-dependent linearization* was developed [7]-[8] so that the linearized model can be updated as the robot moves along the desired path. Based upon the linearization about a moving operating point whose position and orientation are functions of time, this improved method results in a linear, time-varying system which can accurately model the robot dynamics. The study in [3] found that if a time-varying system is lexicografixedly controllable, then there exists a state feedback controller [4] which can assign an arbitrary set of eigenvalues to the closed-loop feedback system. This paper considers the control problem of a robotic system which is modeled by a linear, time-varying system that is not lexicografixedly controllable. A new approach to be proposed will consider discretization of the time-varying system and selection of a sampling time to make the resulting time-varying, discrete system become lexicografixedly controllable. Developed algorithms for lexicografixedly controllable systems given in [4] may then be applied to control the time-varying, discrete, robotic systems.

This paper presents some preliminary results of the study and is organized as follows. Section 2 presents the discretization using zero-order hold (ZOH) devices. Section 3 defines the controllability of the resulting, time-varying, discrete system and introduces a theorem to facilitate the testing for controllability. The concept of controllability is then investigated for a single-input-single-output (SISO) system and the results are presented in Section 4. Section 5 outlines the future research and concludes the paper.

2 Discretization

The dynamics of a robot manipulator having n degrees of freedom can be described by the following equation of motion,

$$\tau(t) = M(q)q'' + N(q,q') + G(q), \tag{1}$$

where $\tau(t)$ is the (nx1) joint torque vector, and q, $q'$and q'' are the (nx1) vectors of joint positions, velocities, and accelerations, respectively. M(q), N(q,q') and G(q) represent the (nxn) inertia matrix, the (nx1) Coriolis

(centrifugal and frictional force) vectors, and the (nx1) gravity vector, respectively.

Linearizing (1) with respect to an operating point whose joint position, velocity, and acceleration are evaluated along a desired path, and are thus functions of time, the following linear, time-varying, state equations are obtained:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t), \end{aligned} \tag{2}$$

where $z(t) = \left| q(t) \quad \dot{q}(t) \right|, y(t) = q(t), u(t) = \tau(t)$, and A(t), B(t), and C(t) are time-varying matrices.

Now if the inputs to the above system are sampled with ZOH devices, as shown in Figure 1,

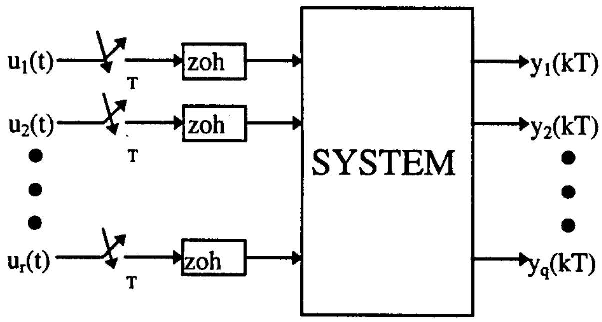

Figure 1 - Linear Time-Varying Multivariable Discrete System

then the resulting system is a linear, time-varying, discrete system described by

$$\begin{aligned} x((k+1)T) &= F(k,T)x(kT) + G(k,T)u(kT) \\ y(kT) &= C(kT)x(kT). \end{aligned} \tag{3}$$

The state and input matrices of the discrete system are defined by

$$F(k,T) = \Phi((k+1)T, kT) \tag{4}$$

and

$$G(k,T) = \int_{kT}^{(k+1)T} \Phi((k+1)T, \tau)B(\tau)d\tau, \tag{5}$$

where $\Phi(t, t_0)$ is the state transition matrix of the continuous system (2). Methods of computing the state transition matrix are well known and can be found in [6].

If the resulting discretized system is determined to be uniformly controllable, uniformly observable, and lexicografixed, then optimization and stabilization methods, similar to those in [2]-[5], should be possible.

There are several obstacles preventing the application of this scheme. The concepts of controllability, observability, and lexicografixedness have not been developed for time-varying, discrete systems. In addition, the methods in [2]-[5] apply only to continuous systems. Similar methods need to be developed for discrete systems. These obstacles are the subject of on-going research. Results in the area of controllability will be presented in the following section.

3 Controllability of Linear Time-Varying Multivariable Discrete Systems

In section 3.1, a controllability matrix for linear, time-varying, multivariable, discrete systems is derived. The utility of this matrix will become apparent in section 3.2, where a controllability theorem for this class of systems is introduced and proved.

3.1 Derivation of the Controllability Matrix

Consider a linear, time-varying, multivariable, discrete, control system whose state equation is defined as

$$x((k+1)T) = F(k,T)x(kT) + G(k,T)u(kT), \tag{6}$$

where

x(kT) = (nx1) state vector at the kth sampling instant
u(kT) = (rx1) control vector at the kth sampling instant
F(k,T) = (nxn) state matrix at the kth sampling instant
G(k,T) = (nxr) input matrix at the kth sampling instant
T = the sampling period.

Definition: The discrete-time control system given by (6) is said to be completely state controllable, if there exists a piecewise-constant control vector u(kT) defined over a finite number of sampling periods, such that starting from any initial state the state x(kT) can be transferred to the desired state x_f in at most n sampling periods [6].

Using the previous definition, a controllability matrix for a linear, time-varying, multivariable, discrete system will now be derived. By recursion, the solution to the system, x(kT), can be found. That is:

For k=0,

$$x(T) = F(0,T)x(0) + G(0,T)u(0). \tag{7}$$

For k=1,

$$\begin{aligned} x(2T) = {} & F(1,T)F(0,T)x(0) + F(1,T)G(0,T)u(0) \\ & + G(1,T)u(T). \end{aligned} \tag{8}$$

For k=2,

$$\begin{aligned} x(3T) = {} & F(2,T)F(1,T)F(0,T)x(0) \\ & + F(2,T)F(1,T)G(0,T)u(0) \\ & + F(2,T)G(1,T)u(T) + G(2,T)u(2T). \end{aligned} \tag{9}$$

For k=3,

$$\begin{aligned} x(4T) = {} & F(3,T)F(2,T)F(1,T)F(0,T)x(0) \\ & + F(3,T)F(2,T)F(1,T)G(0,T)u(0) \\ & + F(3,T)F(2,T)G(1,T)u(T) \\ & + F(3,T)G(2,T)u(2T) + G(3,T)u(3T). \end{aligned} \tag{10}$$

Repeating this procedure, gives

$$\begin{aligned} & x(kT) = [F(k-1,T)F(k-2,T)\ldots F(0,T)]x(0) \\ & + \sum_{j=0}^{k-1} [[F(k-1,T)F(k-2,T)\ldots F(j+1,T)]G(jT)u(jT). \end{aligned} \tag{11}$$

If k=n, the following equation is obtained.

$$\begin{aligned} & x(nT) = [F(n-1,T)F(n-2,T)\ldots F(0,T)]x(0) \\ & + \sum_{j=0}^{n-1} [[F(n-1,T)F(n-2,T)\ldots F(j+1,T)]G(jT)u(jT) \end{aligned} \tag{12}$$

The previous equation can be rewritten as

$$\begin{aligned} & x(nT) - [F(n-1,T)F(n-2,T)\ldots F(0,T)]x(0) \\ & = \sum_{j=0}^{n-1} [F(n-1,T)F(n-2,T)\ldots F(j+1,T)]G(j,T)u(jT) \\ & = [F(n-1,T)F(n-2,T)\ldots F(1,T)]G(0,T)u(0) \\ & + [F(n-1,T)F(n-2,T)\ldots F(2,T)]G(1,T)u(T) \\ & + [F(n-1,T)F(n-2,T)\ldots F(3,T)]G(2,T)u(2T) \\ & + \ldots + G(n-1,T)u((n-1)T). \end{aligned} \tag{13}$$

Equation (13) can be written in a matrix form as

$$x(nT) - [F(n-1,T)F(n-2,T)\ldots F(0,T)]x(0) = \begin{vmatrix} G(n-1,T) \\ F(n-1,T)G(n-2,T) \\ \vdots \\ [F(n-1,T)F(n-2,T)\ldots F(1,T)]G(0,T) \end{vmatrix}^T \begin{vmatrix} u((n-1)T) \\ u((n-2)T) \\ \vdots \\ u(0) \end{vmatrix}. \tag{14}$$

Consequently, a controllability matrix can be defined as

$$S(T) = \begin{vmatrix} G(n-1,T) \\ F(n-1,T)G(n-2,T) \\ \vdots \\ [F(n-1,T)F(n-2,T)\ldots F(1,T)]G(0,T) \end{vmatrix}^T. \tag{15}$$

Substituting (15) into (14) yields,

$$\begin{aligned} & x(nT) - [F(n-1,T)F(n-2,T)\ldots F(0,T)]x(0) \\ & = S(T) \begin{vmatrix} u((n-1)T) \\ u((n-2)T) \\ \vdots \\ u(0) \end{vmatrix}. \end{aligned} \tag{16}$$

In compact form, the controllability matrix can be written as

$$S(T) = | S_0(T) \quad S_1(T) \quad \ldots \quad S_{n-1}(T) |, \tag{17}$$

where

$$S_0(T) = G(n-1,T), \tag{18}$$

and

$$S_i(T) = \left[\prod_{j=1}^{i} F(n-j,T)\right] G(n-1-i,T). \tag{19}$$

Since G(k,T)is an (nxr) matrix, each of the matrices $S_0(T)$, $S_1(T)$,...,$S_{n-1}(T)$ is an (nxr) matrix. Therefore, the controllability matrix is an (nxnr) matrix.

3.2 Controllability Theorem

Testing for the state controllability of a linear, time-varying, multivariable, discrete system is greatly facilitated by the following theorem.

Theorem: The state equation (6) of a linear, time-varying, multivariable, discrete, system is completely state controllable, if and only if the rank of the (nxnr) controllability matrix is n at some sampling period, T. The condition for complete state controllability is

$$\text{rank}(S(T)) = n. \tag{20}$$

Proof: The proof is similar to that of linear, time-invariant, discrete systems [6]. To prove the necessity, assume the state equation is state controllable, and then show that the rank of the controllability matrix equals n. The state transition equation of the discrete system was written in (16) as

$$x(nT) - [F(n-1,T)F(n-2,T)\ldots F(0,T)]x(0) = S(T)\begin{vmatrix} u((n-1)T) \\ u((n-2)T) \\ \vdots \\ u(0) \end{vmatrix}. \tag{21}$$

The left hand side of the previous equation can be represented as an (nx1) vector, X(nT). Then (21) can be written as

$$X(nT) = S(T)U(nT). \tag{22}$$

If the system is assumed to be completely state controllable, then every initial state, X(0), can be transferred by unconstrained controls, U(nT), to any final state, X(nT), for finite N. Thus the problem is that of
given S(T) and every vector X(nT) in the n-dimensional state space, solve for the controls U(nT). Since (22) represents n simultaneous linear equations, from the theory of linear equations, these equations must be linearly independent for solutions to exist. Therefore the matrix S(T) must have full rank (equal to n) for solutions to exist.

Contradiction is used to prove the sufficiency of the theorem. Assume that the controllability matrix does not have full rank, and that the system is completely state controllable. If

$$\text{rank}(S(T)) < n, \tag{23}$$

then, from the theory of linear equations, S(T) does not have n linearly independent columns. If S(T) does not have at least n linearly independent columns, then given X(nT) and U(nT), the set of simultaneous linear equations, given in (22) as

$$X(nT) = S(T)U(nT), \tag{24}$$

cannot be solved.

This means that unconstrained controls to transfer every initial state X(0) to any final state X(nT) cannot be found. This contradicts the assumption that the system is completely state controllable. Consequently, the rank condition given by (20) is found to be a necessary and sufficient condition for complete state controllability. The proof is now complete.

4 Example

This section contains an example to illustrate the concepts presented in the previous two sections. In this example, it is assumed that the fixed-point linearization scheme has been applied to a non-linear robotic system and that the linearization point is allowed to vary. The result is a linear, time-varying, continuous system with dynamic equations given by (2). Assume that the state equation of the resulting system is given by

$$\dot{x}(t) = \begin{vmatrix} -1 & \exp(2t) \\ 0 & -1 \end{vmatrix} x(t) + \begin{vmatrix} t^2 \\ 0 \end{vmatrix} u(t). \tag{25}$$

The controllability matrix can be computed as

$$Q(t) = \begin{vmatrix} t^2 & t^2+2t \\ 0 & 0 \end{vmatrix}. \tag{26}$$

It is obvious that the determinant of the controllability matrix is zero. This implies that the system is not uniformly controllable. Therefore the methods in [2]-[5], cannot be used to control the system.

However, the scheme introduced in section 2 can be used to control the system. The system is discretized by the application of a sample and zero-order hold to the input. If this is done, the discretization equations (4)-(5) must be applied to the system. The state equation of the resulting linear, time-varying, discrete system is given by (6). The resulting state and input matrices are computed as

$$F(k,T) = \begin{vmatrix} \exp(-T) & \exp\left(\frac{1}{2}\exp(2(k+1)T - \frac{1}{2}\exp(2kT)\right) \\ 1 & \exp(-T) \end{vmatrix} \tag{27}$$

and

$$G(k,T) = \begin{vmatrix} \begin{pmatrix} (-k^2T^2+2kT-2)\exp(-T) \\ +(k^2+2k+1)T^2-(2k+2)T+2 \end{pmatrix} \\ \left(\frac{1}{3}(k+1)^3-\frac{1}{3}k^3\right)T^3 \end{vmatrix}. \tag{28}$$

From (17)-(19), the controllability matrix, for a single-input system in which the state matrix has dimension n=2, is

$$S(T) = \left|\, G(1,T) \quad F(1,T)G(0,T) \,\right|. \tag{29}$$

Solving for the controllability matrix, yields

$$S(T) = \begin{vmatrix} \begin{matrix} (+2T-T^2-2)\exp(-T) \\ +4T^2-4T+2 \end{matrix} & \begin{matrix} -2\exp(-2T) \\ +(T^2-2T+2)\exp(-T) \\ +\frac{T^3}{3}\exp\begin{pmatrix} \frac{1}{2}\exp(4T) \\ -\frac{1}{2}\exp(2T) \end{pmatrix} \end{matrix} \\ \frac{7}{3}T^3 & \begin{matrix} \left(\frac{1}{3}T^3-2\right)\exp(-T) \\ +T^2-2T+2 \end{matrix} \end{vmatrix}. \tag{30}$$

The theorem presented in section 3, states that a discrete system is completely state controllable, if and only if the controllability matrix has full rank for some sampling period. It is also well known that the controllability matrix has full rank at values of T for which the determinant of the controllability matrix is non- zero. Solving for the determinant, yields

$$\begin{aligned} S_{det}(T) = &-\left(\frac{1}{3}T^5-\frac{2}{3}T^4-4T^3-2T^2+4T-4\right)\exp(-2T) \\ &-\left(T^5-\frac{7}{3}T^4+16T^2-16T+8\right)\exp(-T) \\ &-\frac{7}{9}T^6\exp\left(\frac{1}{2}\exp(4T)-\frac{1}{2}\exp(2T)\right) \\ &+4T^4-12T^3+18T^2-12T+4. \end{aligned} \tag{31}$$

The above equation is plotted as a function of the sampling period.

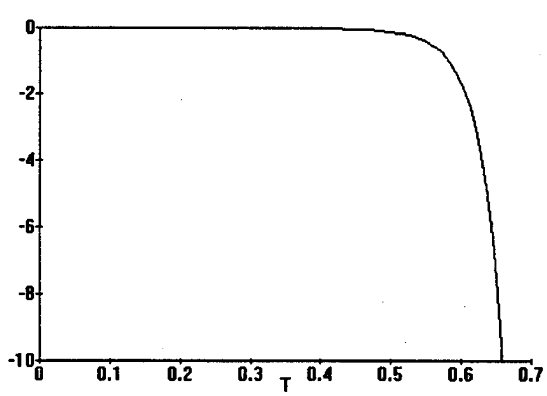

Figure 2 - Determinant as a Function of Sampling Period

Figure (2), shows that the value of the determinant starts out at zero and then decreases. Upon close inspection, it can be seen that the determinant is non-zero for T>0.32. Additionally, solving the previous equation shows that the determinant approaches infinity for T>0.89. Since a determinant of infinity is also undesirable, it can be concluded that the system is completely state controllable for sampling periods in range of

$$0.32 < T < 0.89. \tag{32}$$

Now a sampling period will be selected and it will be shown that, given an initial state, the system can be transformed into a final desired state in n sampling periods. If a sampling period of T=0.5 seconds is selected, the state equation becomes

$$\begin{vmatrix} x_0(k+1) \\ x_1(k+1) \end{vmatrix} = \begin{vmatrix} \exp\left(-\frac{1}{2}\right) & \exp\begin{pmatrix} \frac{1}{2}\exp(k+1) \\ -\frac{1}{2}\exp(k) \end{pmatrix} \\ 1 & \exp\left(-\frac{1}{2}\right) \end{vmatrix} \begin{vmatrix} x_0(k) \\ x_1(k) \end{vmatrix} + \begin{vmatrix} \begin{pmatrix} \left(-\frac{1}{4}k^2 + k - 2\right)\exp\left(-\frac{1}{2}\right) \\ +\frac{1}{4}k^2 - \frac{1}{2}k + \frac{5}{4} \end{pmatrix} \\ \frac{1}{8}k^2 + \frac{1}{8}k + \frac{1}{24} \end{vmatrix} u(k). \tag{33}$$

If the initial state, at k=0, is given as

$$\begin{vmatrix} x_0(0) \\ x_1(0) \end{vmatrix} = \begin{vmatrix} 2 \\ 5 \end{vmatrix}, \tag{34}$$

then the state equation, at k=0, becomes

$$\begin{vmatrix} x_0(1) \\ x_1(1) \end{vmatrix} = \begin{vmatrix} \begin{pmatrix} 2\exp\left(-\frac{1}{2}\right) + 5\exp\left(\frac{1}{2}\exp(1) - \frac{1}{2}\right) \\ +\left(\frac{5}{4} - 2\exp\left(-\frac{1}{2}\right)\right)u(0) \end{pmatrix} \\ 2 + 5\exp\left(-\frac{1}{2}\right) + \frac{1}{24}u(0) \end{vmatrix}. \tag{35}$$

The state equation, at k=1, becomes

$$\begin{vmatrix} x_0(2) \\ x_1(2) \end{vmatrix} = \begin{vmatrix} \left(-\frac{5}{4}\exp\left(-\frac{1}{2}\right) + 1\right)u(1) \\ +\exp\left(\frac{1}{2}\exp(2) - \frac{1}{2}\exp(1)\right)\left(2 + 5\exp\left(-\frac{1}{2}\right)\right) \\ +\begin{pmatrix} -2\exp(-1) + \frac{5}{4}\exp\left(-\frac{1}{2}\right) \\ +\frac{1}{24}\exp\left(\frac{1}{2}\exp(2) - \frac{1}{2}\exp(1)\right) \end{pmatrix} u(0) \\ +\exp\left(-\frac{1}{2}\right)\left(2\exp\left(-\frac{1}{2}\right) + 5\exp\left(\frac{1}{2}\exp(1) - \frac{1}{2}\right)\right) \\ \frac{7}{24}u(1) + \left(-\frac{47}{24}\exp\left(-\frac{1}{2}\right) + \frac{5}{4}\right)u(0) + 2\exp\left(-\frac{1}{2}\right) \\ +5\exp\left(\frac{1}{2}\exp(1) - \frac{1}{2}\right) + \exp\left(-\frac{1}{2}\right)\left(2 + 5\exp\left(-\frac{1}{2}\right)\right) \end{vmatrix}. \tag{36}$$

If the desired state is given as

$$\begin{vmatrix} x_0(2) \\ x_1(2) \end{vmatrix} = \begin{vmatrix} \frac{1}{2} \\ \frac{5}{2} \end{vmatrix}, \tag{37}$$

then it can be shown, by substituting (37) into (36) and then solving the set of simultaneous linear equations, that inputs of

$$u(0) = -119.956 \quad \text{and} \quad u(1) = -20.944 \tag{38}$$

will produce the desired state (37).

5 Conclusions

This paper presents the development of a new robotic control scheme. The scheme involves linearizing, through fixed-point linearization, and discretizing, through sample and hold techniques, the robotic system. Through the application of these methods, a robotic system can be modeled as a linear, time-varying, multivariable, discrete system.

Until now, the concept of controllability had not been developed for such systems. In this paper, the controllability of linear, time-varying, multivariable,

discrete systems was investigated. Using the definition of controllability, a controllability matrix was derived and a controllability theorem introduced. It was proved that the full rank condition of the controllability matrix is a necessary and sufficient condition for the system to be completely state controllable. Through an example, it was shown that, if the system is controllable, the state variables of a linear, time-varying, discrete system can be transformed to any arbitrary state in a finite number of sampling periods.

On-going research focuses on expanding these results to the observability property of such systems and then exploiting the uses of the controllability and observability matrices. State feedback controllers and state estimators are two potential applications. Once these concepts have been fully developed for linear, time-varying, multivariable, discrete systems, then the implementation of the control scheme presented in this paper can be applied to control non-linear robotic systems.

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